

DUAL SPACES TO ORLICZ-LORENTZ SPACES

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ABSTRACT. For an Orlicz function φ and a decreasing weight w , two intrinsic exact descriptions are presented for the norm in the Köthe dual of an Orlicz-Lorentz function space $\Lambda_{\varphi,w}$ or a sequence space $\lambda_{\varphi,w}$, equipped with either Luxemburg or Amemiya norms. The first description of the dual norm is given via the modular $\inf\{\int \varphi_*(f^*/|g|)|g| : g \prec w\}$, where f^* is the decreasing rearrangement of f , $g \prec w$ denotes the submajorization of g by w and φ_* is the complementary function to φ . The second one is stated in terms of the modular $\int_I \varphi_*((f^*)^0/w)w$, where $(f^*)^0$ is Halperin's level function of f^* with respect to w . That these two descriptions are equivalent results from the identity $\inf\{\int \psi(f^*/|g|)|g| : g \prec w\} = \int_I \psi((f^*)^0/w)w$ valid for any measurable function f and Orlicz function ψ . Analogous identity and dual representations are also presented for sequence spaces.

1. INTRODUCTION

The main goal of the paper is to give an isometric description of the Köthe dual space of Orlicz-Lorentz space $\Lambda_{\varphi,w}$, where φ is an Orlicz function and w is a decreasing locally integrable weight function. The Orlicz-Lorentz spaces have been studied extensively for the past two decades, since when their basic properties were established in [7]. So far however there have not been given satisfactory isometric description of the dual spaces of Orlicz-Lorentz spaces. There are several different isomorphic representations of the Köthe dual spaces $(\Lambda_{\varphi,w})'$ given for example in [6] or in [8]. In [3] is posted an unsolved problem number XIV asking for finding an isometric representation of $(\Lambda_{\varphi,w})'$.

Orlicz-Lorentz spaces can be treated as a special case of more general Calderón-Lozanovskii spaces. Lozanovskii in his paper [18] (see also [13]-[17], [20] and [21]) proved a duality theorem, which in particular can be applied to Orlicz-Lorentz spaces. However his original formulas are too general and not explicit enough for applications in the setting of Lorentz type spaces. Here we show that Lozanovskii's formulas for dual norms and the Köthe dual spaces can be expressed in terms of the recently introduced modular $P_{\varphi,w}$ and the corresponding modular space $\mathcal{M}_{\varphi,w}$ (see [9]). In fact $\mathcal{M}_{\varphi,w} = \{f \in L^0 : P_{\varphi,w}(\lambda f) < \infty \text{ for some } \lambda > 0\}$, where L^0 is the space of Lebesgue measurable real functions on $I = [0, \alpha)$ and

$$P_{\varphi,w}(f) = \inf \left\{ \int_I \varphi(f^*/|g|) |g| : g \prec w \right\}.$$

The notation $g \prec w$ means that g is submajorized by w , that is $\int_0^t g^* \leq \int_0^t w$ for all $t \in I$.

In the case when $\varphi(u) = u^p$, $1 < p < \infty$, and $\Lambda_{\varphi,w}$ becomes a classical Lorentz space $\Lambda_{p,w}$, a different explicit isometric description of its dual was given by Halperin in [4]. He introduced the notion of level intervals and level functions with respect to the weight w , and applied them to obtain the formula of the norm of dual space. Here we study the level functions and the modulars in the environment of Orlicz-Lorentz spaces, which allows us to extend Halperin's theorem to the case of those spaces.

¹2010 *Mathematics Subject Classification*: 42B25, 46B10, 46E30

²*Key words and phrases*: Orlicz-Lorentz spaces, Lorentz spaces, dual spaces, level function, Calderón-Lozanovskii spaces, r.i. spaces

Consequently we give in this paper two different isometric representations of dual spaces of Orlicz-Lorentz spaces, one by means of submajorization by the weight w , and another one by level functions with respect to w . They are valid for both function and sequence spaces.

The paper is organized as follows. In section 1 we give basic notations and notions needed further. Among others we define Calderón-Lozanovskii spaces and Orlicz-Lorentz spaces equipped with standard Amemiya and Luxemburg norms.

In section 2 we recall the definition of function spaces $\mathcal{M}_{\varphi,w}$ and then applying the general duality theorem of Lozanovskii, we prove that the Köthe dual space $(\Lambda_{\varphi,w})'$ is $\mathcal{M}_{\varphi^*,w}$ with equality of corresponding norms. In case when the space $\Lambda_{\varphi,w}$ is separable, it is also an isometric representation of its dual space. This representation is given for both Amemiya and Luxemburg norms.

Section 3 is devoted to a number of specific properties of the modular $P_{\varphi,w}(f)$. There is given a sequence of technical results that leads to the main theorem describing an algorithm for calculation of infimum in the formula of the modular $P_{\varphi,w}(f)$ when f is a simple decreasing function. This is Theorem 3.9 which states that the function g^f produced by Algorithm A is minimizing the modular $P_{\varphi,w}(f)$. It is interesting to observe that g^f depends only on f and w , but not on φ .

In section 4 we give another isometric representation of the Köthe dual spaces using the so called level functions f^0 with respect to w that Halperin introduced in [4]. Applying the results of the previous section, in particular Algorithm A, we first prove that $P_{\varphi,w}(f) = \int_I \varphi(f/g^f)g^f = \int_{\varphi}(f^0/w)w$ for a decreasing simple function f . In the next step we extend this result to any $f \in \Lambda_{\varphi,w}$, which in fact yields the second duality theorem. Theorem 4.8 summarizes all Köthe duality formulas for function spaces $\Lambda_{\varphi,w}$ equipped with either Amemiya or Luxemburg norms. Halperin's duality result for spaces $\Lambda_{p,w}$, $1 < p < \infty$, is then a corollary from Theorem 4.8.

In the last fifth section we present the analogous results for the Orlicz-Lorentz sequence spaces $\lambda_{\varphi,w}$. We show first that sequence spaces as well as their Köthe dual spaces can be embedded isometrically into appropriate Orlicz-Lorentz function spaces. Next applying the results of the previous sections for function spaces we quickly obtain the analogous isometric representations of the dual spaces of $\lambda_{\varphi,w}$ in terms of the sequence spaces $\mathfrak{m}_{\varphi^*,w}$ introduced in [9] as well as in terms of the spaces generated by φ^* , w and level sequences.

Let us agree first on the notation and basic notions used in the paper. By φ we denote an *Orlicz function*, that is $\varphi : [0, \infty) \rightarrow [0, \infty)$, $\varphi(0) = 0$, φ is convex and φ is strictly increasing. Let φ_* be the *complementary* function to φ , that is $\varphi_*(s) = \sup_{t \geq 0} \{st - \varphi(t)\}$, $s \geq 0$. By φ^{-1} denote the inverse function to φ . It is said that φ is an *N-function* whenever $\lim_{t \rightarrow 0+} \varphi(t)/t = 0$ and $\lim_{t \rightarrow \infty} \varphi(t)/t = \infty$. It is well known that φ_* is an *N-function* whenever φ is such a function [10]. Recall also that the function $t \mapsto \varphi(a/t)t$ is decreasing and convex on \mathbb{R}_+ for every $a > 0$. The first fact results from the well known property that the function $\varphi(t)/t$ is increasing for $t > 0$, while for the second one we have by convexity of φ that for any $t_1, t_2 \geq 0$,

$$\varphi\left(\frac{2a}{t_1 + t_2}\right)\frac{t_1 + t_2}{2} = \varphi\left(\frac{at_1}{(t_1 + t_2)t_1} + \frac{at_2}{(t_1 + t_2)t_2}\right)\frac{t_1 + t_2}{2} \leq \frac{t_1}{2}\varphi\left(\frac{a}{t_1}\right) + \frac{t_2}{2}\varphi\left(\frac{a}{t_2}\right).$$

It also shows that $t \mapsto \varphi(a/t)t$ is strictly convex if φ is strictly convex. It is said that φ satisfies a Δ_2 -condition for all arguments, respectively for large arguments, whenever $\varphi(2u) \leq K\varphi(u)$ for all $u \geq 0$, respectively for all $u \geq u_0$ and some $u_0 \geq 0$.

Given an Orlicz function φ , define its associated *Calderón-Lozanovskii function* as

$$(1.1) \quad \rho(t, s) = \rho_{\varphi}(t, s) = \varphi^{-1}(s/t)t, \quad s \geq 0, \quad t > 0,$$

and the *conjugate function* to ρ as

$$\hat{\rho}(t, s) = \hat{\rho}_{\varphi}(t, s) = \inf_{u, v > 0} \frac{us + vt}{\rho(u, v)}, \quad s, t \geq 0.$$

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It is well known that the function $\rho_\varphi(t, s)$ is concave on $(0, \infty) \times [0, \infty)$. Moreover, if φ is an N -function then

$$(1.2) \quad \hat{\rho}_\varphi(t, s) = \varphi_*^{-1}(t/s) s, \quad t \geq 0, \quad s > 0,$$

(see Example 3 in [17], or Example 7 in [21]).

Let further $I = [0, \alpha)$ where $0 < \alpha \leq \infty$. By L^0 denote the set of all Lebesgue measurable real-valued functions on I . Given $f \in L^0$ define its *distribution function* as

$$d_f(\lambda) = \mu\{t \in I : |f(t)| > \lambda\}, \quad \lambda \geq 0,$$

and its *decreasing rearrangement* f^* as

$$f^*(t) = \inf\{\lambda > 0 : d_f(\lambda) \leq t\}, \quad t \in I.$$

Here by decreasing or increasing functions we mean the functions which are non-increasing or non-decreasing, respectively. We say that $f \in L^0$ is *submajorized* by $g \in L^0$ and we write

$$f \prec g \quad \text{whenever} \quad \int_0^t f^* \leq \int_0^t g^* \quad \text{for every } t \in I.$$

For any decreasing locally integrable function h let further incorporate the following notation

$$H(t) = \int_0^t h, \quad t \in I.$$

A Banach space $(E, \|\cdot\|_E)$ is called a *Banach function space* (or a *Köthe space*) if $E \subset L^0$ and whenever $f \in L^0$, $g \in E$ and $|f| \leq |g|$ a.e. then $f \in E$ and $\|f\|_E \leq \|g\|_E$. We will also assume that each Banach function space contains a weak unit, i.e. there is $f \in E$ such that $f(t) > 0$ for a.a. $t \in I$. By E' denote the Köthe dual space to E , which consists of all $f \in L^0$ such that $\|f\|_{E'} = \sup\{\int_I fg : \|g\|_E \leq 1\} < \infty$. The space E' equipped with the norm $\|\cdot\|_{E'}$ is a Banach function space. It is well known that E' is non-trivial and contains a weak unit [24, Ch.15, §71, Theorem 4(a)].

Given a Calderón-Lozanovskii function ρ and a couple of Banach function spaces E, F , the *Calderón-Lozanovskii space* is defined as

$$\rho^0(E, F) = \{f \in L^0 : \|f\|_\rho^0 = \inf\{\|g\|_E + \|h\|_F : |f| = \rho(|g|, |h|)\} < \infty\},$$

$$\rho(E, F) = \{f \in L^0 : \|f\|_\rho = \inf\{\max(\|g\|_E, \|h\|_F) : |f| = \rho(|g|, |h|)\} < \infty\}.$$

Recall that the spaces $\rho_\varphi^0(L^\infty, L^1)$ and $\rho_\varphi(L^\infty, L^1)$ coincide isometrically with the Orlicz space L^φ equipped with its *Amemiya* and *Luxemburg norm* respectively [18]. Moreover, in the above definitions one may take equivalently $|f| \leq \rho(|g|, |h|)$ instead of $|f| = \rho(|g|, |h|)$. In fact it is enough to apply Lemma 1 from [21], which states that if $\|g\|_E \leq 1, \|h\|_F \leq 1$ such that $0 \leq f \leq \rho(g, h)$, then there exist $0 \leq g_1 \leq g, 0 \leq f_1 \leq f$ satisfying $f = \rho(g_1, h_1)$. It is also known [19] that

$$\begin{aligned} \|f\|_{\rho(E, F)} &= \inf\{C > 0 : |f| \leq C\rho(|g|, |h|), \|g\|_E \leq 1, \|h\|_F \leq 1\} \\ &= \inf\{C > 0 : |f| = C\rho(|g|, |h|), \|g\|_E \leq 1, \|h\|_F \leq 1\}. \end{aligned}$$

Both spaces $\rho(E, F)$ and $\rho^0(E, F)$ coincide as sets and the norms $\|\cdot\|_\rho, \|\cdot\|_\rho^0$ are evidently equivalent. The spaces $\hat{\rho}(E, F), \hat{\rho}^0(E, F)$ are defined analogously as $\rho(E, F)$ and $\rho^0(E, F)$ where the function ρ is replaced by $\hat{\rho}$. Moreover, the notation $\rho_\varphi(E, F)$ stands for the function ρ that is defined by φ according to formula (1.1).

Let w be a *weight function* on I that is $w \in L^0$, w is positive and decreasing on I , and locally integrable, i.e. $W(t) = \int_0^t w < \infty, t \in I$. Denote $W(\infty) = \int_0^\infty w$ in case when $\alpha = \infty$. The *Lorentz space* Λ_w is classically defined as

$$\Lambda_w = \left\{f \in L^0 : \|f\|_{\Lambda_w} = \int_I f^* w = \int_I f^* dW < \infty\right\},$$

and the *Marcinkiewicz space* M_W as

$$M_W = \left\{ f \in L^0 : \|f\|_{M_W} = \sup_{t \in I} \left(\int_0^t f^*/W(t) \right) < \infty \right\}.$$

Both are Banach function spaces and each one is the Köthe dual space of the other one. Note that $\|f\|_{M_W} \leq 1$ if and only if $f \prec w$. Let φ be an Orlicz function and w a decreasing weight function on I . Then the *Orlicz-Lorentz space* is the set

$$\Lambda_{\varphi,w} = \{f \in L^0 : \exists_{\lambda > 0} I_{\varphi,w}(\lambda f) < \infty\},$$

where $I_{\varphi,w}(f) = \int_I \varphi(f^*)w$. It is equipped with either the Luxemburg norm

$$\|f\|_{\Lambda} = \inf\{\epsilon > 0 : I_{\varphi,w}(f/\epsilon) \leq 1\},$$

or the Amemiya norm

$$\|f\|_{\Lambda}^0 = \inf_{k > 0} \frac{1}{k} (1 + I_{\varphi,w}(kf)).$$

By $\Lambda_{\varphi,w}$ we denote the Orlicz-Lorentz space equipped with the Luxemburg norm $\|\cdot\|_{\Lambda}$, and by $\Lambda_{\varphi,w}^0$ this same space equipped with the Amemiya norm $\|\cdot\|_{\Lambda}^0$. The Orlicz-Lorentz spaces are Calderón-Lozanovskii spaces relative to the couple (L^∞, Λ_w) , and the following identities

$$(1.3) \quad \Lambda_{\varphi,w} = \rho_{\varphi}(L^\infty, \Lambda_w), \quad \Lambda_{\varphi,w}^0 = \rho_{\varphi}^0(L^\infty, \Lambda_w)$$

hold true with equalities of norms. The first equality may be found in [19] (cf. [5] and [6]). As for the second one letting $f \in \rho_{\varphi}^0(L^\infty, E)$, we have

$$(1.4) \quad \begin{aligned} \|f\|_{\rho_{\varphi}(L^\infty, E)}^0 &= \inf \{ \|x\|_E + \|y\|_{L^\infty} : |x| = |y|\varphi(|f|/|y|) \} \\ &= \inf_{k > 0} \{ \inf \{ \| |y|\varphi(|f|/|y|) \|_E + \|y\|_{L^\infty} : \|y\|_{L^\infty} = k \} \} \\ &= \inf_{k > 0} \{ \|k\varphi(|f|/k)\|_E + k \} = \inf_{k > 0} \frac{1}{k} (\|\varphi(k|f|)\|_E + 1), \end{aligned}$$

where the third equality is a consequence of the inequality $|y| \leq \|y\|_{L^\infty}$ and the monotonicity of the function $s \mapsto s\varphi(a/s)$. The desired equality follows for $E = \Lambda_w$.

2. THE DUAL SPACE OF AN ORLICZ-LORENTZ SPACE

In this section we will show that the Köthe dual spaces to the Orlicz-Lorentz spaces $\Lambda_{\varphi,w}$ and $\Lambda_{\varphi,w}^0$ coincide isometrically with the spaces $\mathcal{M}_{\varphi^*,w}^0$ and $\mathcal{M}_{\varphi^*,w}$, respectively. The spaces $\mathcal{M}_{\varphi,w}$ have been recently introduced in the paper [9]. Given an Orlicz function φ and a weight w let

$$\mathcal{M}_{\varphi,w} = \{f \in L^0 : \exists_{\lambda > 0} P_{\varphi,w}(f/\lambda) < \infty\},$$

where the modular $P_{\varphi,w}$ is defined as

$$P_{\varphi,w}(f) = \inf \left\{ \int_I \varphi\left(\frac{f^*}{|g|}\right) |g| : g \prec w \right\} = \inf \left\{ \left\| \varphi\left(\frac{f^*}{|g|}\right) g \right\|_1 : g \prec w \right\}.$$

Here and further in the paper by $\|\cdot\|_1$ we denote the norm in the space L^1 of integrable functions on I . In order to avoid any ambiguity in the definition of the modular $P_{\varphi,w}$ let us agree on the convention that for any measurable functions $f, g \geq 0$ on I , if $g(t) = 0$ then

$$\varphi\left(\frac{f(t)}{g(t)}\right)g(t) = \begin{cases} 0 & \text{if } f(t) = 0; \\ \infty & \text{if } f(t) \neq 0. \end{cases}$$

It is also worth to observe that

$$(2.1) \quad P_{\varphi,w}(f) = \inf \left\{ \left\| \varphi\left(\frac{f^*}{|g|}\right) g \right\|_1 : \|g\|_w = 1 \right\}.$$

In fact by convexity of φ one has $\frac{1}{a}\varphi(at) \leq \varphi(t)$ for each $t > 0$ and $0 < a \leq 1$. Therefore, if $\|g\|_{M_W} = a < 1$ then

$$\left\| \varphi\left(\frac{af^*}{k|g|}\right) \frac{kg}{a} \right\|_1 \leq \left\| \varphi\left(\frac{f^*}{k|g|}\right) kg \right\|_1.$$

We introduce two equivalent norms on $\mathcal{M}_{\varphi,w}$. The first one is of the Luxemburg type,

$$\|f\|_{\mathcal{M}} = \|f\|_{\mathcal{M}_{\varphi,w}} = \inf\{\lambda > 0 : P_{\varphi,w}(f/\lambda) \leq 1\},$$

and the second one is of the Amemiya type,

$$\|f\|_{\mathcal{M}}^0 = \|f\|_{\mathcal{M}_{\varphi,w}}^0 = \inf_{k>0} \frac{1}{k} (P_{\varphi,w}(kf) + 1).$$

By $\mathcal{M}_{\varphi,w}$ denote the space equipped with the norm $\|\cdot\|_{\mathcal{M}}$ and by $\mathcal{M}_{\varphi,w}^0$ the space endowed with the norm $\|\cdot\|_{\mathcal{M}}^0$. The first result expresses the spaces $\mathcal{M}_{\varphi,w}$ and $\mathcal{M}_{\varphi,w}^0$ as Calderón-Lozanovskii spaces relative to the couple (M_W, L_1) .

Proposition 2.1. *Let φ be an N -function. Then*

$$\begin{aligned} \rho_{\varphi}(M_W, L^1) &= \mathcal{M}_{\varphi,w} \quad \text{and} \quad \rho_{\varphi}^0(M_W, L^1) = \mathcal{M}_{\varphi,w}^0, \\ \hat{\rho}_{\varphi}(L^1, M_W) &= \mathcal{M}_{\varphi^*,w} \quad \text{and} \quad \hat{\rho}_{\varphi}^0(L^1, M_W) = \mathcal{M}_{\varphi^*,w}^0, \end{aligned}$$

with equalities of corresponding norms.

Proof. Let $f \in \rho_{\varphi}(M_W, L^1)$. Then

$$\begin{aligned} \|f\|_{\rho_{\varphi}(M_W, L^1)} &= \|f^*\|_{\rho_{\varphi}(M_W, L^1)} = \inf \left\{ \max\{\|g\|_{M_W}, \|h\|_1\} : f^* = \rho_{\varphi}(|g|, |h|) \right\} \\ &= \inf \left\{ C > 0 : f^* = C\rho_{\varphi}(|g|, |h|), \|g\|_{M_W} \leq 1, \|h\|_1 \leq 1 \right\} \\ &= \inf \left\{ C > 0 : \varphi\left(\frac{f^*}{C|g|}\right) |g| = |h|, \|g\|_{M_W} \leq 1, \|h\|_1 \leq 1 \right\} \\ &= \inf \left\{ C > 0 : \left\| \varphi\left(\frac{f^*}{C|g|}\right) g \right\|_1 \leq 1, \|g\|_{M_W} \leq 1 \right\} \\ &= \inf \left\{ C > 0 : \inf \left\{ \left\| \varphi\left(\frac{f^*}{C|g|}\right) g \right\|_1 : g \prec w \right\} \leq 1 \right\} = \|f\|_{\mathcal{M}_{\varphi,w}}. \end{aligned}$$

Applying (2.1) we also get the second of the first two equalities

$$\begin{aligned} \|f\|_{\rho_{\varphi}(M_W, L^1)}^0 &= \|f^*\|_{\rho_{\varphi}(M_W, L^1)}^0 = \inf \left\{ \|g\|_{M_W} + \|h\|_1 : f^* = \rho_{\varphi}(|g|, |h|) \right\} \\ &= \inf \left\{ \|g\|_{M_W} + \left\| \varphi\left(\frac{f^*}{|g|}\right) g \right\|_1 : g \in M_W \right\} \\ &= \inf_{k>0} \left\{ \inf \left\{ k + \left\| \varphi\left(\frac{f^*}{k|g|}\right) kg \right\|_1 : \|g\|_{M_W} = 1 \right\} \right\} \\ &= \inf_{k>0} \left\{ \inf \left\{ k + \left\| \varphi\left(\frac{f^*}{k|g|}\right) kg \right\|_1 : \|g\|_{M_W} \leq 1 \right\} \right\} \\ &= \inf_{k>0} \frac{1}{k} (P_{\varphi,w}(kf) + 1) = \|f\|_{\mathcal{M}_{\varphi,w}}^0. \end{aligned}$$

The remaining equalities are proved analogously by (1.2). \square

Now we are ready to state an isometric characterization of the (Köthe) dual spaces of Orlicz-Lorentz spaces.

Theorem 2.2. *Let w be a decreasing weight and φ be an N -function. Then the following holds true.*

(1) *The Köthe dual spaces to Orlicz-Lorentz spaces $\Lambda_{\varphi,w}$ and $\Lambda_{\varphi,w}^0$ are expressed as*

$$(\Lambda_{\varphi,w})' = \mathcal{M}_{\varphi^*,w}^0 \quad \text{and} \quad (\Lambda_{\varphi,w}^0)' = \mathcal{M}_{\varphi^*,w},$$

with equality of corresponding norms.

(2) *Let φ satisfy the appropriate Δ_2 -condition, that is (i) for large arguments if $I = [0, \alpha)$ with $\alpha < \infty$, or $\alpha = \infty$ and $W(\infty) < \infty$; (ii) for all arguments if $I = [0, \infty)$ and $W(\infty) = \infty$. Then the dual spaces $(\Lambda_{\varphi,w})^*$ and $(\Lambda_{\varphi,w}^0)^*$ are isometrically isomorphic to their corresponding Köthe dual spaces. In fact for any functional $\Phi \in (\Lambda_{\varphi,w})^*$ (resp., $\Phi \in (\Lambda_{\varphi,w}^0)^*$) there exists $\phi \in \mathcal{M}_{\varphi^*,w}^0$ (resp., $\phi \in \mathcal{M}_{\varphi^*,w}$) such that*

$$\Phi(f) = \int_I f \phi, \quad f \in \Lambda_{\varphi,w},$$

and $\|\Phi\|_{(\Lambda_{\varphi,w})^} = \|\phi\|_{\mathcal{M}_{\varphi^*,w}^0}$ (resp., $\|\Phi\|_{(\Lambda_{\varphi,w}^0)^*} = \|\phi\|_{\mathcal{M}_{\varphi^*,w}}$).*

Proof. By Lozanovskii's representation theorem [18] for any Banach function spaces E, F we have

$$(\rho(E, F))' = \hat{\rho}^0(E', F') \quad \text{and} \quad (\rho^0(E, F))' = \hat{\rho}(E', F'),$$

with norm equalities. Notice that $(\Lambda_w)' = M_W$. This fact was proved in [11, Theorem 5.2, p. 112] under the assumption that $W(\infty) = \infty$ in case of $I = [0, \infty)$, however the same proof works for any decreasing locally integrable weight function w . Thus by (1.2), (1.3) and Proposition 2.1 we get

$$(\Lambda_{\varphi,w})' = (\rho(L^\infty, \Lambda_w))' = \hat{\rho}^0(L^1, M_W) = \mathcal{M}_{\varphi^*,w}^0.$$

The second equality can be shown analogously. The second part of the hypothesis follows from the well known fact that the Orlicz-Lorentz spaces are order continuous [7] under the assumption of the appropriate Δ_2 -condition, and the general theorem stating that the Köthe dual space E' of an order continuous Banach function space E is isometrically isomorphic via integral functionals to the dual space E^* [1, Theorem 4.1, p. 20]. \square

3. AN ALGORITHM FOR COMPUTING $P_{\varphi,w}(f)$ FOR A DECREASING SIMPLE FUNCTION f

In this section our goal is to find a function g which minimizes the formula $P_{\varphi,w}(f)$ for a given simple decreasing function $f = \sum_{i=1}^n a_i \chi_{[t_{i-1}, t_i)}$ with $a_1 > a_2 > \dots > a_n > 0$ and $0 = t_0 < t_1 < \dots < t_n < \infty$. This process consists of several steps and leads to an algorithm which reveals that such a function g exists and depends only on f and w , but not on φ .

First in Lemma 3.1 we show that the minimizing function g has to be also simple and decreasing. In the second step in Lemma 3.3 we show that such a minimizing function g exists. Next, in Lemma 3.4 it is proved that $G(t_n) = W(t_n)$ and then in Theorem 3.7 is demonstrated that $g = \sum_{j=0}^{m-1} \lambda_j f \chi_{[t_{i_j}, t_{i_{j+1}})}$ for some $(\lambda_i)_{i=0}^{m-1}$, $(t_{i_j})_{j=0}^{m-1}$ and $m \leq n$, where $W(t_{i_j}) = G(t_{i_j})$. This shows that g needs to be piecewise proportional to f and the ratios λ_j are determined by the points t_{i_j} . Therefore in order to find g it is sufficient to determine points t_{i_j} . This process will be described by Algorithm A. Applying finally Theorem 3.7 and Lemma 3.6, we finish with proving that Algorithm A produces the function g that minimizes $P_{\varphi,w}(f)$.

Lemma 3.1. *If $f = \sum_{i=1}^n a_i \chi_{A_i}$ where $a_1 > \dots > a_n > 0$ and $A_i = [t_{i-1}, t_i)$ with $0 = t_0 < t_1 < \dots < t_n < \infty$, then*

$$(3.1) \quad P_{\varphi,w}(f) = \inf \left\{ \begin{array}{l} \|\varphi(\frac{f}{g})g\|_1 : g \prec w, \\ \text{and } g = \sum_{i=1}^n b_i \chi_{A_i} \text{ with } b_1 \geq b_2 \geq \dots \geq b_n > 0 \end{array} \right\}.$$

Proof. Let $f = \sum_{i=1}^n a_i \chi_{A_i}$ satisfy the assumptions. Corollary 4.5 in [9] states that

$$P_{\varphi,w}(f) = \inf\{\|\varphi(f/g)g\|_1 : g \prec w, 0 \leq g \downarrow\},$$

where $g \downarrow$ means that g is decreasing. Fix some $g \prec w$, $g \downarrow$ and put

$$h = \varphi\left(\frac{f}{g}\right)g \text{ and } \tilde{h} = \varphi\left(\frac{f}{Tg}\right)Tg,$$

where

$$T : g \mapsto \sum_{i=1}^n \left(\frac{1}{|A_i|} \int_{A_i} g \right) \chi_{A_i}.$$

Since g is decreasing Tg is also decreasing. Therefore it is enough to show that $\|\tilde{h}\|_1 \leq \|h\|_1$ and $Tg \prec w$.

By Proposition 3.7 in [1, Chap. 2] we have $Tg \prec g$ and so $Tg \prec w$. By convexity of the function $s \mapsto \varphi(a/s)s$, $a > 0$, and Jensen's inequality for convex functions we have for every $i = 1, \dots, n$,

$$\varphi\left(\frac{a_i}{(1/|A_i|) \int_{A_i} g}\right) \frac{1}{|A_i|} \int_{A_i} g \leq \frac{1}{|A_i|} \int_{A_i} \varphi\left(\frac{a_i}{g}\right)g,$$

which gives

$$\|\tilde{h}\|_1 = \sum_{i=1}^n |A_i| \varphi\left(\frac{a_i}{(1/|A_i|) \int_{A_i} g}\right) \frac{1}{|A_i|} \int_{A_i} g \leq \sum_{i=1}^n \int_{A_i} \varphi\left(\frac{a_i}{g}\right)g = \|h\|_1,$$

and the proof is finished. \square

Lemma 3.2. Suppose that $g = g^* = \sum_{i=1}^n b_i \chi_{[t_{i-1}, t_i)}$ where $0 = t_0 < t_1 < \dots < t_n < \infty$. Then

$$\inf_{0 < t \leq t_n} \frac{W(t)}{G(t)} = \min_{i=1, \dots, n} \frac{W(t_i)}{G(t_i)}.$$

In particular $g \prec w$ if and only if $G(t_i) \leq W(t_i)$ for each $i = 1, \dots, n$.

Proof. The left-hand side is clearly majorized by the right-hand one. Conversely if for some $\theta \geq 0$,

$$W(t_i) \geq \theta G(t_i), \quad i = 1, \dots, n,$$

which remains trivially true for $i = 0$, then by concavity of W and the fact that G is affine on each segment $[t_i, t_{i+1}]$, we have for every $\lambda \in [0, 1]$ and $i = 0, \dots, n-1$,

$$\begin{aligned} W((1-\lambda)t_i + \lambda t_{i+1}) &\geq (1-\lambda)W(t_i) + \lambda W(t_{i+1}) \\ &\geq (1-\lambda)\theta G(t_i) + \lambda \theta G(t_{i+1}) = \theta G((1-\lambda)t_i + \lambda t_{i+1}), \end{aligned}$$

which proves the converse inequality. \square

Lemma 3.3. Let φ be an N -function. Then for a simple function $f = \sum_{i=1}^n a_i \chi_{A_i}$ such that $a_1 > \dots > a_n > 0$ and $A_i = [t_{i-1}, t_i)$ where $0 = t_0 < t_1 < \dots < t_n < \infty$, there exists $g = \sum_{i=1}^n b_i \chi_{A_i}$ with $b_1 \geq \dots \geq b_n > 0$, $\|g\|_{M_W} = 1$ and such that $P_{\varphi,w}(f) = \|\varphi(f/g)g\|_1$. Consequently in the definition of $P_{\varphi,w}(f)$ the infimum is attained.

Proof. In fact, by Lemma 3.1 the infimum in the definition of $P_{\varphi,w}(f)$ may be considered over $g = \sum_{i=1}^n b_i \chi_{A_i}$ with $b_1 \geq \dots \geq b_n > 0$ such that $g \prec w$. Applying now Lemma 3.2 the condition $g \prec w$ is equivalent to $G(t_k) = \sum_{i=1}^k b_i |A_i| \leq W(t_k)$ for each $k = 1, \dots, n$. But those constraints define the set

$$C = \left\{ b = (b_1, \dots, b_n) : b_1 \geq \dots \geq b_n > 0, \sum_{i=1}^k b_i |A_i| \leq W(t_k), k = 1, \dots, n \right\},$$

which is relatively compact in \mathbb{R}^n . Hence if the sequence $(b^k) = ((b_1^k, \dots, b_n^k))_{k=1}^\infty \subset \mathbb{R}^n$ is minimizing for the infimum in formula (3.1) in the definition of $P_{\varphi, w}(f)$, there is a subsequence (b^{k_j}) such that $b^{k_j} \rightarrow b \in \bar{C}$. Let us show that $b \in C$. If for the contrary $b \in \bar{C} \setminus C$ then $b_i = 0$ for some $i = 1, \dots, n$, which means that $b_i^{k_j} \rightarrow 0$ if $j \rightarrow \infty$. Setting $g_{k_j} = \sum_{i=1}^n b_i^{k_j} \chi_{A_i}$, by the fact that φ is an N -function we get for $j \rightarrow \infty$,

$$\left\| \varphi\left(\frac{f}{g_{k_j}}\right) g_{k_j} \right\|_1 \geq \varphi\left(\frac{a_i}{b_i^{k_j}}\right) b_i^{k_j} |A_i| \rightarrow \infty,$$

which ensures that (b^{k_j}) cannot be minimizing for the infimum in the definition of $P_{\varphi, w}(f)$, a contradiction. \square

Example. We present an example which shows that for decreasing simple functions f the functions g that minimize $P_{\varphi, w}(f)$ depend on f .

Let $\varphi(t) = t^2$, $w(t) = 1/2\sqrt{t}$, $t > 0$. Define the family of functions $f_x := x\chi_{(0,1)} + 1\chi_{(1,4)}$ on $(0, \infty)$ for $x \geq 1$. Then by Lemmas 3.1 – 3.4,

$$P_{\varphi, w}(f_x) = \min \left\{ \begin{array}{l} \frac{x^2}{b_1} + \frac{3}{b_2} : g = b_1\chi_{[0,1)} + b_2\chi_{[1,4)}, \\ \text{with } 1 \geq b_1 \geq b_2 \text{ and } b_1 + 3b_2 = 2 \end{array} \right\}.$$

Applying Lagrange multipliers method to minimize the function $\psi(b_1, b_2) = \frac{x^2}{b_1} + \frac{3}{b_2}$ with constraint $b_1 + 3b_2 = 2$ gives the solution $b_1 = \frac{2x}{x+3}$, $b_2 = \frac{2}{x+3}$. We have $b_1 \geq b_2$ for $x \geq 1$. Moreover, if $1 \leq x \leq 3$ then $b_1 \leq 1$. If $x \geq 3$ then there is no extremum in the set defined by constraints $1 \geq b_1 \geq b_2$ and $b_1 + 3b_2 = 2$ and therefore ψ attains its minimum at $(1, 1/3)$ or $(1/2, 1/2)$. Finally we get

$$P_{\varphi, w}(f_x) = \begin{cases} \frac{x(x+3)}{2} + \frac{3(x+3)}{2} & \text{for } 1 \leq x \leq 3, \\ x^2 + 9 & \text{for } x \geq 3, \end{cases}$$

and it is clear that g cannot be chosen independently of f_x .

Lemma 3.4. Let φ be an N -function. Let $f = \sum_{i=1}^n a_i \chi_{A_i}$ for some a_i with $a_1 > \dots > a_n > 0$ and $A_i = [t_{i-1}, t_i)$ where $0 = t_0 < t_1 < \dots < t_n < \infty$. If $g = \sum_{i=1}^n b_i \chi_{A_i} = g^*$ is a minimizing function for $P_{\varphi, w}(f)$ then

$$(3.2) \quad G(t_n) = \int_0^{t_n} g = \int_0^{t_n} w = W(t_n).$$

Proof. One may assume that $\|g\|_{M_W} = 1$. By Lemma 3.3 we also have that $b_1 \geq \dots \geq b_n > 0$, so $g > 0$ on $[0, t_n)$. Suppose g does not satisfy (3.2) that is $G(t_n) < W(t_n)$. We will then find a new function h such that $h \prec w$ and $\|\varphi(f/h)h\|_1 < \|\varphi(f/g)g\|_1$ contradicting the minimality of g .

By Lemma 3.2, $\|g\|_{M_W} = 1$ is equivalent to

$$1 = \sup_{t_n \geq t > 0} \frac{G(t)}{W(t)} = \inf_{t_n \geq t > 0} \frac{W(t)}{G(t)} = \min_{i=1, \dots, n} \frac{W(t_i)}{G(t_i)}.$$

It follows that $\{i > 0 : W(t_i) = G(t_i)\} \neq \emptyset$. Let

$$i_1 = \max\{i > 0 : W(t_i) = G(t_i)\} \quad \text{and} \quad \gamma_1 = \min_{i_1 < i \leq n} \left\{ \frac{W(t_i) - W(t_{i_1})}{G(t_i) - G(t_{i_1})} \right\}.$$

Since $G(t_n) < W(t_n)$ we have $i_1 < n$. Then $W(t_i) > G(t_i)$ for all $i > i_1$, and thus it is clear that $\gamma_1 > 1$. Set

$$g_1 = g\chi_{[0, t_{i_1})} + \gamma_1 g\chi_{[t_{i_1}, t_n)}.$$

Note that $g_1 = g_1^*$. In fact, since g is decreasing, it is sufficient to show that $b_{i_1} \geq \gamma_1 b_{i_1+1}$. We have $G(t_{i_1}) = G_1(t_{i_1}) = W(t_{i_1})$ and $G_1(t_{i_1+1}) \leq W(t_{i_1+1})$ by definition of γ_1 . Then

$$w(t_{i_1})(t_{i_1+1} - t_{i_1}) \geq W(t_{i_1+1}) - W(t_{i_1}) \geq G_1(t_{i_1+1}) - G_1(t_{i_1}) = \gamma_1 b_{i_1+1}(t_{i_1+1} - t_{i_1}).$$

On the other hand $G(t_{i_1-1}) = G_1(t_{i_1-1}) \leq W(t_{i_1-1})$, so

$$b_{i_1}(t_{i_1} - t_{i_1-1}) = G_1(t_{i_1}) - G_1(t_{i_1-1}) \geq W(t_{i_1}) - W(t_{i_1-1}) \geq w(t_{i_1})(t_{i_1} - t_{i_1-1}).$$

Therefore $b_{i_1} \geq w(t_{i_1}) \geq \gamma_1 b_{i_1+1}$. We also have that $g_1 \prec w$. Indeed in view of Lemma 3.2 and definition of i_1 it is enough to check the inequality $G_1(t_i) \leq W(t_i)$ for $i > i_1$. We have

$$\begin{aligned} G_1(t_i) &= G(t_{i_1}) + \gamma_1(G(t_i) - G(t_{i_1})) \\ &\leq G(t_{i_1}) + \frac{W(t_i) - W(t_{i_1})}{G(t_i) - G(t_{i_1})}(G(t_i) - G(t_{i_1})) = W(t_i). \end{aligned}$$

However, $\|\varphi(f/g_1)g_1\|_1 < \|\varphi(f/g)g\|_1$ in view of $g_1 \geq g$ and $g_1 \neq g$ and the fact that $\varphi(a/t)t$ is a strictly decreasing function of t for each $a > 0$ (by the assumption that φ is an N -function). This contradicts the fact that g realizes the infimum in the definition of $P_{\varphi,w}(f)$. \square

Lemma 3.5. *Let φ be an N -function. Let $f = \sum_{i=1}^n a_i \chi_{A_i}$ for some a_i with $a_1 > \dots > a_n > 0$, $A_i = [t_{i-1}, t_i]$ where $0 = t_0 < t_1 < \dots < t_n < \infty$, and $g = \sum_{i=1}^n b_i \chi_{A_i} = g^*$ be a minimizing function for $P_{\varphi,w}(f)$. Assume also that for some $k = 1, \dots, n-1$ we have $G(t_k) = W(t_k)$. Then $g\chi_{[0,t_k]}$ is a minimizing function for $P_{\varphi,w}(f\chi_{[0,t_k]})$ that is*

$$\|\varphi(f/g)g\chi_{[0,t_k]}\|_1 = P_{\varphi,w}(f\chi_{[0,t_k]}),$$

while

$$(3.3) \quad \|\varphi(f/g)g\chi_{[t_k,t_n]}\|_1 = \inf \left\{ \sum_{i=k+1}^n \varphi(a_i/c_i)c_i|A_i| : c_{k+1} > \dots > c_n > 0, \sum_{i=k+1}^j c_i|A_i| \leq W(t_j) - W(t_k), k < j \leq n \right\}.$$

Proof. By Lemma 3.3 we have that $b_1 \geq \dots \geq b_n > 0$ and by $g \prec w$ and $G(t_k) = W(t_k)$ it holds

$$\sum_{i=k+1}^j b_i|A_i| \leq W(t_j) - W(t_k), \quad k < j \leq n.$$

Note that the problem of minimizing (3.3) is equivalent to minimizing $P_{\varphi,w_k}(f_k)$, where for $t \in I$ we let

$$f_k(t) = (f\chi_{[t_k,t_n]})(t + t_k), \quad w_k(t) = (w\chi_{[t_k,t_n]})(t + t_k).$$

By Lemma 3.3 applied to the interval $[0, t_n - t_k]$ there is a simple, decreasing function $h^{(1)} = \sum_{i=k+1}^n h_i \chi_{A_i - t_k} \prec w_k$ that minimizes $P_{\varphi,w_k}(f_k)$. On the other hand by Lemma 3.3 and Lemma 3.4 applied to the interval $[0, t_k]$ there is a simple, decreasing function $h^{(2)} = \sum_{i=1}^k h_i \chi_{A_i} \prec w\chi_{[0,t_k]}$ that minimizes $P_{\varphi,w}(f\chi_{[0,t_k]})$, and it holds that $H^{(2)}(t_k) = \int_0^{t_k} h^{(2)} = W(t_k)$. Thus

$$(3.4) \quad \sum_{i=1}^k \varphi(a_i/h_i)h_i|A_i| = P_{\varphi,w}(f\chi_{[0,t_k]}) \leq \sum_{i=1}^k \varphi(a_i/b_i)b_i|A_i|,$$

$$(3.5) \quad \sum_{i=k+1}^n \varphi(a_i/h_i)h_i|A_i| = P_{\varphi,w_k}(f_k) \leq \sum_{i=k+1}^n \varphi(a_i/b_i)b_i|A_i|,$$

and so

$$(3.6) \quad \sum_{i=1}^n \varphi(a_i/h_i)h_i|A_i| \leq \sum_{i=1}^n \varphi(a_i/b_i)b_i|A_i|.$$

Now let $h := \sum_{i=1}^n h_i \chi_{A_i}$ and note that h is decreasing. Indeed we will show that $h_k \geq h_{k+1}$. Since $H(t_{k-1}) = H^{(2)}(t_{k-1}) \leq W(t_{k-1})$, and $H(t_k) = H^{(2)}(t_k) = W(t_k)$ we have

$$h_k |A_k| = H(t_k) - H(t_{k-1}) \geq W(t_k) - W(t_{k-1}) \geq w_k |A_k|,$$

while

$$\begin{aligned} h_{k+1} |A_{k+1}| &= H(t_{k+1}) - H(t_k) = h_{k+1} |A_{k+1}| = H^{(1)}(t_{k+1} - t_k) \\ &\leq W_k(t_{k+1} - t_k) = W(t_{k+1}) - W(t_k) \leq w_k |A_{k+1}|, \end{aligned}$$

and thus $h_k \geq w_k \geq h_{k+1}$. Note also that $h \prec w$, since $H(t_i) \leq W(t_i)$, $i = 1, \dots, k$, by $h^{(2)} \prec w$, $H(t_k) = H^{(2)}(t_k) = W(t_k)$, and $H(t_i) - H(t_k) \leq W(t_i) - W(t_k)$, $i = k+1, \dots, n$, by $h^{(1)} \prec w_k$. Then

$$\sum_{i=1}^n \varphi(a_i/h_i) h_i |A_i| \geq P_{\varphi, w}(f) = \sum_{i=1}^n \varphi(a_i/b_i) b_i |A_i|.$$

Consequently we have equality in (3.6) and also in both inequalities (3.4), (3.5), and thus the proof is completed. \square

Lemma 3.6. *Let φ be a strictly convex N -function. Let $f = \sum_{i=1}^n a_i \chi_{A_i}$ for some a_i with $a_1 > \dots > a_n > 0$ and $A_i = [t_{i-1}, t_i)$ where $0 = t_0 < t_1 < \dots < t_n < \infty$ and $g = \sum_{i=1}^n b_i \chi_{A_i}$ with $b_1, \dots, b_n > 0$. If $g \neq \lambda f$ for $\lambda = \frac{G(t_n)}{F(t_n)}$ then*

$$\|\varphi(f/g)g\|_1 > \|\varphi(f/[\lambda f])\lambda f\|_1.$$

Proof. Set $\lambda_i = b_i/a_i$, $i = 1, \dots, n$. If all the λ_i are equal to, say, λ' then $g = \lambda' \sum_{i=1}^n a_i \chi_{A_i} = \lambda' f$ and $G(t_n) = \lambda' F(t_n)$ thus $\lambda' = \lambda$. If $g \neq \lambda f$, then not all λ_i are equal and by Jensen's inequality and strict convexity of φ it holds

$$\begin{aligned} \frac{1}{G(t_n)} \left\| \varphi\left(\frac{f}{g}\right)g \right\|_1 &= \sum_{i=1}^n \varphi\left(\frac{1}{\lambda_i}\right) \frac{\lambda_i a_i |A_i|}{G(t_n)} > \varphi\left(\sum_{i=1}^n \frac{1}{\lambda_i} \frac{\lambda_i a_i |A_i|}{G(t_n)}\right) \\ &= \varphi\left(\frac{1}{\lambda}\right) = \frac{1}{G(t_n)} \varphi\left(\frac{1}{\lambda}\right) \|\lambda f\|_1 = \frac{1}{G(t_n)} \left\| \varphi\left(\frac{f}{\lambda f}\right) \lambda f \right\|_1. \end{aligned}$$

\square

Theorem 3.7. *Let φ be a strictly convex N -function. Let $f = \sum_{i=1}^n a_i \chi_{A_i}$ for some a_i with $a_1 > \dots > a_n > 0$ and $A_i = [t_{i-1}, t_i)$ where $0 = t_0 < t_1 < \dots < t_n < \infty$. If $g = \sum_{i=1}^n b_i \chi_{A_i} = g^*$ is a simple function realizing the infimum in the definition of $P_{\varphi, w}(f)$ then g has to be of the form*

$$(3.7) \quad g = \sum_{j=0}^{m-1} \lambda_j f \chi_{[t_{i_j}, t_{i_{j+1}})},$$

for some λ_j , $j = 0, 1, \dots, m-1$, where $0 = i_0 < i_1 < \dots < i_m = n$ and

$$G(t_{i_j}) = W(t_{i_j})$$

for each $j = 0, 1, \dots, m$.

Proof. Let g satisfy the assumptions of the theorem. By Lemma 3.3 we have that $b_1 \geq \dots \geq b_n > 0$ and by Lemma 3.4 that $G(t_n) = W(t_n)$. Define a finite sequence (i_0, i_1, \dots, i_m) as

$$i_0 = 0 \quad \text{and} \quad i_j = \min \{i > i_{j-1} : G(t_i) = W(t_i)\}, \quad j = 1, \dots, m.$$

As a consequence of applying $(m-1)$ times Lemma 3.5 to f , that is decomposing f first as $f \chi_{[0, t_{i_{m-1}})} + f \chi_{[t_{i_{m-1}}, t_n]}$, then $f \chi_{[0, t_{i_{m-1}})}$ as $f \chi_{[0, t_{i_{m-2}})} + f \chi_{[t_{i_{m-2}}, t_{i_{m-1}})}$, etc., we obtain that

if $g = g^* = \sum_{i=1}^n b_i \chi_{A_i}$ minimizes $P_{\varphi, w}(f)$ and $G(t_{i_j}) = W(t_{i_j})$, $G(t_{i_{j+1}}) = W(t_{i_{j+1}})$, then $(b_{i_j+1}, \dots, b_{i_{j+1}})$ also minimizes the sum

$$\sum_{i=i_j+1}^{i_{j+1}} \varphi(a_i/b_i) b_i |A_i|$$

under the constraints $B_j(k) := \sum_{i=i_j+1}^k b_i |A_i| \leq W(t_k) - W(t_{i_j})$ for $k = i_j + 1, \dots, i_{j+1}$.

Therefore we may consider each interval $[t_{i_j}, t_{i_{j+1}})$, $j = 0, 1, \dots, m-1$, separately and we will show that $g\chi_{[t_{i_j}, t_{i_{j+1}})} = \lambda_j f\chi_{[t_{i_j}, t_{i_{j+1}})}$ where

$$\lambda_j = \frac{W(t_{i_{j+1}}) - W(t_{i_j})}{F(t_{i_{j+1}}) - F(t_{i_j})}.$$

If $t_{i_{j+1}} = t_{i_j+1}$ then

$$g\chi_{[t_{i_j}, t_{i_{j+1}})} = \frac{F(t_{i_{j+1}}) - F(t_{i_j})}{t_{i_{j+1}} - t_{i_j}} \lambda_j \chi_{[t_{i_j}, t_{i_{j+1}})} = \lambda_j f\chi_{[t_{i_j}, t_{i_{j+1}})}.$$

If $t_{i_{j+1}} > t_{i_j+1}$ then for all $t_{i_j} < t_i < t_{i_{j+1}}$ one has $G(t_i) < W(t_i)$. In this case consider the function $\psi_j : \mathbb{R}_+^{i_{j+1}-i_j} \rightarrow \mathbb{R}_+$ defined for $b = (b_{i_j+1}, \dots, b_{i_{j+1}})$ as

$$\psi_j(b) = \psi_j(b_{i_j+1}, \dots, b_{i_{j+1}}) = \sum_{i=i_j+1}^{i_{j+1}} \varphi(a_i/b_i) b_i |A_i|,$$

and define the set

$$C_j = \left\{ b \in \mathbb{R}_+^{i_{j+1}-i_j} : b_{i_j+1} \geq b_{i_j+2} \geq \dots \geq b_{i_{j+1}} > 0, \right. \\ \left. \begin{array}{l} \forall_{i_j+1 \leq k < i_{j+1}} B_j(k) < W(t_k) - W(t_{i_j}), \\ B_j(i_{j+1}) = W(t_{i_{j+1}}) - W(t_{i_j}) \end{array} \right\}.$$

Notice that the condition

$$B_j(k) = \sum_{i=i_j+1}^k b_i |A_i| < W(t_k) - W(t_{i_j}), \quad k = i_j + 1, \dots, i_{j+1} - 1,$$

is a consequence of the relation $g \prec w$ and definition of i_j and i_{j+1} . In fact, by Lemma 3.2, $g \prec w$ is equivalent to $G(t_i) \leq W(t_i)$ for each $i = 1, \dots, n$ and by definition of i_j and i_{j+1} we have $G(t_k) < W(t_k)$ for each $k = i_j + 1, \dots, i_{j+1} - 1$. It follows that for $k = i_j + 1, \dots, i_{j+1} - 1$,

$$G(t_{i_j}) + \sum_{i=i_j+1}^k b_i |A_i| = G(t_k) < W(t_k) = W(t_{i_j}) + W(t_k) - W(t_{i_j}),$$

and so $\sum_{i=i_j+1}^k b_i |A_i| < W(t_k) - W(t_{i_j})$.

We need to show now that ψ_j attains its minimum over C_j at the point $\lambda_j a$, $a = (a_{i_j+1}, \dots, a_{i_{j+1}})$.

Consider first the simplex $S_j = \mathbb{R}_+^{i_{j+1}-i_j} \cap H_j$, where H_j is the hyperplane in $\mathbb{R}^{i_{j+1}-i_j}$ given by the equation $\sum_{i=i_j+1}^{i_{j+1}} |A_i| x_i = W(t_{i_{j+1}}) - W(t_{i_j})$. Then Lemma 3.6 tells us that $\lambda_j a$ is the unique minimizer of ψ_j over S_j . It remains to show that $\lambda_j a \in C_j \subset S_j$. Suppose for the contradiction that $\lambda_j a \notin C_j$. On the other hand, by the previous reasoning, there exists $\bar{b} \in C_j$ that minimizes ψ_j over C_j . Define $b(\lambda) = \lambda \bar{b} + (1 - \lambda) \lambda_j a$ for $0 \leq \lambda \leq 1$. Then by Lemma 3.6 and since $\lambda_j a \neq \bar{b}$ we get $\psi_j(\bar{b}) > \psi_j(\lambda_j a)$. Moreover the strict convexity of $t \mapsto \varphi(d/t)t$ for each $d > 0$, implies strict convexity of ψ_j . Therefore for each $0 < \lambda < 1$,

$$\psi_j(b(\lambda)) < \lambda \psi_j(\bar{b}) + (1 - \lambda) \psi_j(\lambda_j a) < \psi_j(\bar{b}).$$

Notice that for every $0 < \lambda < 1$, $b_{i_j+1}(\lambda) > \dots > b_{i_j+1}(\lambda)$, and

$$\sum_{i=i_j+1}^{i_{j+1}} b_i(\lambda)|A_i| = \lambda \sum_{i=i_j+1}^{i_{j+1}} \bar{b}_i|A_i| + (1-\lambda)\lambda_j(F(t_{i_{j+1}}) - F(t_{i_j})) = W(t_{i_{j+1}}) - W(t_{i_j}).$$

Moreover for each $k = i_j + 1, \dots, i_{j+1} - 1$,

$$\begin{aligned} \sum_{i=i_j+1}^k b_i(\lambda)|A_i| &= \lambda \sum_{i=i_j+1}^k \bar{b}_i|A_i| + (1-\lambda)\lambda_j(F(t_k) - F(t_{i_j})) \\ &< \lambda(W(t_k) - W(t_{i_j})) + (1-\lambda)\lambda_j(F(t_k) - F(t_{i_j})). \end{aligned}$$

This implies that for $0 < \lambda < 1$ sufficiently close to 1, $b(\lambda) \in C_j$. Since $\psi_j(b(\lambda)) < \psi_j(\bar{b})$, the element \bar{b} cannot minimize ψ_j over C_j , which gives the desired contradiction. We have shown therefore that on $[t_{i_j}, t_{i_{j+1}})$ it must be $g = \lambda_j f$ and since j was arbitrary, the proof is finished. \square

The following algorithm will be crucial for proving the main Theorem 3.9 which provides a procedure to obtain a minimizing function g for the modular $P_{\varphi, w}(f)$.

Algorithm A. Let $f = \sum_{i=1}^n a_i \chi_{A_i}$ for some $a_1 > \dots > a_n > 0$ and $A_i = [t_{i-1}, t_i)$ where $0 = t_0 < t_1 < \dots < t_n < \infty$. Define first

$$g_{-1} = f, \quad \gamma_0 = \lambda_0 = \min_{1 \leq i \leq n} \left\{ \frac{W(t_i)}{F(t_i)} \right\}, \quad g_0 = \gamma_0 f = \lambda_0 f, \quad i_0 = 0.$$

Then for $j > 0$ let

$$\begin{aligned} (3.8) \quad i_j &= \max \left\{ i > i_{j-1} : \gamma_{j-1} = \frac{W(t_i) - W(t_{i_{j-1}})}{G_{j-2}(t_i) - G_{j-2}(t_{i_{j-1}})} \right\}, \\ \gamma_j &= \min_{i_j < i \leq n} \left\{ \frac{W(t_i) - W(t_{i_j})}{G_{j-1}(t_i) - G_{j-1}(t_{i_j})} \right\}, \\ g_j &= g_{j-1} \chi_{[0, t_{i_j})} + \gamma_j g_{j-1} \chi_{[t_{i_j}, t_n)}. \end{aligned}$$

Continue the recurrent step until $i_m = n$ for some m and denote $g^f = g_{m-1}$. Clearly $\gamma_j > 1$ for $j = 1, \dots, m-1$, and

$$g^f = \sum_{j=0}^{m-1} \lambda_j f \chi_{[t_{i_j}, t_{i_{j+1}})}, \quad \lambda_j = \prod_{i=0}^j \gamma_i.$$

Hence $\lambda_0 < \lambda_1 < \dots < \lambda_{m-1}$. We also have for $j = 0, 1, \dots, m-1$,

$$\gamma_j = \frac{W(t_{i_{j+1}}) - W(t_{i_j})}{G_{j-1}(t_{i_{j+1}}) - G_{j-1}(t_{i_j})} = \frac{W(t_{i_{j+1}}) - W(t_{i_j})}{\gamma_{j-1}(G_{j-2}(t_{i_{j+1}}) - G_{j-2}(t_{i_j}))} = \frac{W(t_{i_{j+1}}) - W(t_{i_j})}{\prod_{i=0}^{j-1} \gamma_i (F(t_{i_{j+1}}) - F(t_{i_j}))}.$$

Hence

$$\lambda_j = \prod_{i=0}^j \gamma_i = \frac{W(t_{i_{j+1}}) - W(t_{i_j})}{F(t_{i_{j+1}}) - F(t_{i_j})}.$$

It follows that for each $j = 0, 1, \dots, m-1$,

$$G^f(t_{i_j}) = W(t_{i_j}).$$

Now we will show that $g^f \prec w$. Evidently $g_0 = \gamma_0 f \prec w$. Similarly as in Lemma 3.4 we can show that $g_j = g_j^*$ for each j .

Explaining as in Lemma 3.4 we can show that $g_j = g_j^*$. In fact, since f is decreasing, it is sufficient to show that $\lambda_{j-1}a_{i_j} \geq \lambda_j a_{i_j+1}$ for each $j = 1, \dots, m-1$. Fix $j = 1, \dots, m-1$. We have $G_{j-1}(t_{i_j}) = G_j(t_{i_j}) = W(t_{i_j})$ and $G_j(t_{i_j+1}) \leq W(t_{i_j+1})$ by definition of γ_j . Then

$$w(t_{i_j})(t_{i_j+1} - t_{i_j}) \geq W(t_{i_j+1}) - W(t_{i_j}) \geq G_j(t_{i_j+1}) - G_j(t_{i_j}) = \lambda_j a_{i_j+1}(t_{i_j+1} - t_{i_j}).$$

On the other hand $G_{j-1}(t_{i_j-1}) = G_j(t_{i_j-1}) \leq W(t_{i_j-1})$, so

$$\lambda_{j-1}a_{i_j}(t_{i_j} - t_{i_j-1}) = G_j(t_{i_j}) - G_j(t_{i_j-1}) \geq W(t_{i_j}) - W(t_{i_j-1}) \geq w(t_{i_j})(t_{i_j} - t_{i_j-1}).$$

Therefore $\lambda_{j-1}a_{i_j} \geq w(t_{i_j}) \geq \lambda_j a_{i_j+1}$. It remains to prove that $g_{j-1} \prec w$ implies $g_j \prec w$. By (3.8),

$$g_j = g_{j-1}\chi_{[0, t_{i_j})} + \gamma_j g_{j-1}\chi_{[t_{i_j}, t_n]}.$$

Therefore for $k \leq i_j$,

$$G_j(t_k) = G_{j-1}(t_k) \leq W(t_k).$$

If $k > i_j$, then by definition of γ_j ,

$$G_j(t_k) = G_{j-1}(t_{i_j}) + \gamma_j(G_{j-1}(t_k) - G_{j-1}(t_{i_j})) \leq W(t_{i_j}) + W(t_k) - W(t_{i_j}) = W(t_k),$$

and then by Lemma 3.2 we have $g_j \prec w$, which proves that $g^f \prec w$. It is also worth to notice that since $\lambda_0 < \lambda_1 < \dots < \lambda_{m-1}$, the function f/g^f is decreasing.

Remark 3.8. The function g^f produced by Algorithm A is of the form (3.7), but the sequence (t_{i_j}) obtained in this way need not to be maximal in the sense that there may exist $t_i \notin (t_{i_j})$ such that $G^f(t_i) = \int_0^{t_i} g^f = W(t_i)$.

Now we are ready for our main result describing how to calculate the infimum of $P_{\varphi, w}(f)$ for a decreasing simple function f .

Theorem 3.9. Let φ be an N -function and let $f = \sum_{i=1}^n a_i \chi_{A_i}$ for some a_i with $a_1 > \dots > a_n > 0$ and $A_i = [t_{i-1}, t_i]$ where $0 = t_0 < t_1 < \dots < t_n < \infty$. Then the function g^f produced by Algorithm A is a minimizing function for $P_{\varphi, w}(f)$, that is

$$P_{\varphi, w}(f) = \left\| \varphi\left(\frac{f}{g^f}\right) g^f \right\|_1.$$

The function g^f is independent of φ and depends only on f and w .

Proof. We divide the proof into two parts.

(I) Assume first that φ is strictly convex. Let g^f be produced by Algorithm A. Suppose that a function h is minimizing as in Theorem 3.7. We will prove that $h = g^f$. This will be done by induction on the number s of steps in Algorithm A.

(a) Assume first that $s = 1$, that is $\min_{1 \leq i \leq n} \{W(t_i)/F(t_i)\} = W(t_n)/F(t_n)$. Then $g^f = \lambda_0 f$, with $\lambda_0 = W(t_n)/F(t_n)$. On the other hand by Theorem 3.7, $h = \sum_{j=0}^{p-1} \mu_j f \chi_{[t_{k_j}, t_{k_{j+1}}]}$, where $0 = k_0 < k_1 < \dots < k_p = n$ and $H(t_{k_j}) = W(t_{k_j})$, $j = 1, \dots, p$. But since $\lambda_0 f \prec w$, Lemma 3.6 shows that $\mu_0 = \dots = \mu_{p-1} = \lambda_0$, that is $h = \lambda_0 f = g^f$.

(b) Assume now that $s > 1$ and that Algorithm A is valid for $s-1$ steps. We claim first that

$$(3.9) \quad W(t_{i_1}) = H(t_{i_1}),$$

where $i_1 = \max \{i > 0 : \lambda_0 = W(t_i)/F(t_i)\}$, $\lambda_0 = \min_{1 \leq i \leq n} \{W(t_i)/F(t_i)\}$. Clearly $i_1 < n$. If the claim is false then $H(t_{i_1}) < W(t_{i_1})$. Now since $H(t_i) \leq W(t_i)$ for all $i = 1, \dots, n$, two cases are possible:

- (i) $H(t_i) < W(t_i)$ for each $0 < i \leq i_1$ or,
- (ii) $H(t_k) = W(t_k)$ for some $k < i_1$ and $H(t_{i_1}) < W(t_{i_1})$.

Case (i): Suppose that $H(t_i) < W(t_i)$ for each $i \leq i_1$. Then by (3.7), $h\chi_{[0,t_m]} = \lambda f\chi_{[0,t_m]}$ with $H(t_m) = W(t_m)$ for some $\lambda > 0$ and $t_m > t_{i_1}$. Hence $\lambda F(t_{i_1}) = H(t_{i_1}) < W(t_{i_1}) = \lambda_0 F(t_{i_1})$ and thus $\lambda < \lambda_0$. It follows that

$$H(t_m) = \lambda F(t_m) < \lambda_0 F(t_m) \leq W(t_m),$$

which is a contradiction with $H(t_m) = W(t_m)$.

Case (ii). Suppose that $H(t_{i_1}) < W(t_{i_1})$ and $W(t_k) = H(t_k)$ for some $k < i_1$. Assume that k is the biggest index satisfying those conditions. Since h is assumed to be a minimizing function, by Theorem 3.7 there exist $t_{i_1} < t_m \leq t_n$ and $\lambda > 0$ such that

$$(3.10) \quad h\chi_{[t_k,t_m]} = \lambda f\chi_{[t_k,t_m]} \quad \text{and} \quad H(t_m) = W(t_m).$$

Since $\lambda_0 F \leq W$ and $\lambda_0 F(t_{i_1}) = W(t_{i_1})$. We have

$$\lambda(F(t_{i_1}) - F(t_k)) = H(t_{i_1}) - W(t_k) < W(t_{i_1}) - W(t_k) \leq \lambda_0(F(t_{i_1}) - F(t_k))$$

and

$$\lambda(F(t_m) - F(t_{i_1})) = W(t_m) - H(t_{i_1}) > \lambda_0 F(t_m) - W(t_{i_1}) = \lambda_0(F(t_m) - F(t_{i_1}))$$

which gives the contradiction. Therefore we have shown the claim (3.9).

Next we will show that

$$(3.11) \quad h\chi_{[0,t_{i_1}]} = \lambda_0 f\chi_{[0,t_{i_1}]} = g^f\chi_{[0,t_{i_1}]}.$$

Suppose for the contrary that

$$h\chi_{[0,t_{i_1}]} = \sum_{j=0}^{r-1} \delta_j f\chi_{[t_{k(j)}, t_{k(j+1)}]},$$

where $0 = t_{k(0)} < t_{k(1)} < \dots < t_{k(r)} = t_{i_1}$ with $\delta_j \neq \lambda_0$ for some $j = 0, 1, \dots, r-1$. Then by Lemma 3.6 applied to the interval $[0, t_{i_1}]$ and λ_0 , we have

$$\begin{aligned} \|\varphi(f/h)h\|_1 &= \|\varphi(f/h)h\chi_{[0,t_{i_1}]} \|_1 + \|\varphi(f/h)h\chi_{[t_{i_1}, t_n]} \|_1 > \\ &> \|\varphi(f/(\lambda_0 f))\lambda_0 f\chi_{[0,t_{i_1}]} \|_1 + \|\varphi(f/h)h\chi_{[t_{i_1}, t_n]} \|_1. \end{aligned}$$

It follows that h is not minimizing $P_{\varphi,w}(f)$, which contradicts our assumption and proves (3.11).

Now in view of $H(t_{i_1}) = W(t_{i_1})$ we have by the proof of Lemma 3.5 that $h_{i_1}(t) = h\chi_{[t_{i_1}, t_n]}(t + t_{i_1})$ is a minimizing function in the modular $P_{\varphi, w_{i_1}}(f_{i_1})$ where for $t \in I$,

$$f_{i_1}(t) = (f\chi_{[t_{i_1}, t_n]})(t + t_{i_1}), \quad w_{i_1}(t) = (w\chi_{[t_{i_1}, t_n]})(t + t_{i_1}).$$

On the other hand it is straightforward to see that Algorithm A for f_{i_1} , and the weight w_{i_1} , has $s-1$ steps and that it yields a function $g^{f_{i_1}}$ which is nothing but the function $g^f\chi_{[t_{i_1}, t_n]}$ shifted backward by t_{i_1} , that is

$$g^{f_{i_1}}(t) = g^f(t + t_{i_1}).$$

Now by induction hypothesis we have $g^{f_{i_1}} = h_{i_1}$ and thus

$$g^f\chi_{[t_{i_1}, t_n]} = h\chi_{[t_{i_1}, t_n]}.$$

We know also by Lemma 3.5 that $h\chi_{[0,t_{i_1}]}$ is minimizing $P_{\varphi,w}(f\chi_{[0,t_{i_1}]})$ while clearly Algorithm A for $f\chi_{[0,t_{i_1}]}$ has only one step and yields $g^f\chi_{[0,t_{i_1}]}$. Thus by part (a), $g^f\chi_{[0,t_{i_1}]} = h\chi_{[0,t_{i_1}]}$. Therefore $g^f = h$ and this finishes the proof of case (I).

(II) Assume now that φ is any N -function. Let $\varphi_m(t) = \varphi(t) + \frac{1}{m}t^2$. Then the functions φ_m are strictly convex N -functions and $\varphi_m \rightarrow \varphi$ uniformly on compact sets. Let g^f be produced by Algorithm A. Suppose g^f is not minimizing for $P_{\varphi,w}(f)$, i.e. there is $h = \sum_{i=1}^n b_i \chi_{A_i} \prec w$, such that for some $\delta > 0$ we have

$$\|\varphi(f/h)h\|_1 + \delta \leq \|\varphi(f/g^f)g^f\|_1.$$

Since h , f and g^f are simple functions,

$$\|\varphi_m(f/h)h\|_1 \rightarrow \|\varphi(f/h)h\|_1 \quad \text{and} \quad \|\varphi_m(f/g^f)g^f\|_1 \rightarrow \|\varphi(f/g^f)g^f\|_1.$$

Hence there is N such that for $m > N$,

$$\|\varphi_m(f/h)h\|_1 \leq \|\varphi(f/h)h\|_1 + \delta/3 \quad \text{and} \quad \|\varphi_m(f/g^f)g^f\|_1 \geq \|\varphi(f/g^f)g^f\|_1 - \delta/3.$$

Therefore for each $m > N$,

$$\|\varphi_m(f/h)h\|_1 + \delta/3 \leq \|\varphi_m(f/g^f)g^f\|_1,$$

which means that g^f does not minimize $P_{\varphi_m, w}(f)$ and so it contradicts the case (I) and the proof is completed. \square

4. DUAL NORMS OF $\Lambda_{\varphi, w}$ IN TERMS OF LEVEL FUNCTIONS.

In this section we develop formulas for Köthe duals of Orlicz-Lorentz spaces equipped with Luxemburg or Amemiya norms in terms of level functions. Let w be a weight function on I as defined in introduction. For $f = f^*$ locally integrable on I , define after Halperin [4] for $0 \leq a < b < \infty$, $a, b \in I$,

$$W(a, b) = \int_a^b w, \quad F(a, b) = \int_a^b f, \quad R(a, b) = \frac{F(a, b)}{W(a, b)},$$

and for $b = \infty$,

$$R(a, b) = R(a, \infty) = \limsup_{t \rightarrow \infty} R(a, t).$$

Then $(a, b) \subset I$ is called a *level interval* (resp. *degenerate level interval*) of f with respect to w if $b < \infty$ (resp. $b = \infty$) and for each $t \in (a, b)$,

$$R(a, t) \leq R(a, b) \quad \text{and} \quad 0 < R(a, b).$$

It is easy to see that the restriction $0 < R(a, b)$ ensures that any level interval of $f = f^*$ is in fact included in the support of f^* , and this is the only difference with the original definition from [4]. Level interval can be equivalently considered as open, closed or half-closed. If the weight w is fixed then we will say *level interval of f* , or just l.i. for simplicity. If a level interval is not contained in any larger level interval, then it is called *maximal level interval of f with respect to w* , or just maximal level interval and in short m.l.i. In [4], Halperin proved that maximal level intervals of f with respect to w are pairwise disjoint and unique and therefore there is at most countable number of maximal level intervals.

Remark 4.1. (1) The whole semiaxis $(0, \infty)$ may be a degenerate l.i.. Take for example any weight function w and let $f = w$.

(2) Given any weight w , if a decreasing function f is constant on (a, b) then (a, b) is a l.i. of f with respect to w .

First we make a simple observation that the function $t \mapsto (\int_a^t h)/(t-a)$ is decreasing for $t > a$ whenever h is decreasing and locally integrable. Letting now $f(t) = c$ for $t \in (a, b)$, by the fact that w is decreasing, the inequality $R(a, t) \leq R(a, b)$ on (a, b) is equivalent to $\frac{1}{b-a} \int_a^b w \leq \frac{1}{t-a} \int_a^t w$ on (a, b) .

(3) If w is a constant weight then (a, b) is an l.i. of f if and only if f is constant on (a, b) . Consequently any decreasing function with countable many different values has infinite many m.l.i. with respect of a constant weight.

Let now w be constant on I , and (a, b) be a l.i. of f with respect to w . Therefore $F(a, t)/(t-a) \leq F(a, b)/(b-a)$ on (a, b) , and since f is decreasing we have the equality, that is $F(a, t) =$

$F(a, b)(t - a)/(b - a)$ on (a, b) . Hence $f(t) = F(a, b)/(b - a)$ for all $t \in (a, b)$, and so f is constant on (a, b) .

Definition 4.2. [4] Let $f \in L^0$ be decreasing and locally integrable on I . Then the level function f^0 of f with respect to w is defined as

$$f^0(t) = \begin{cases} R(a_n, b_n)w(t) & \text{for } t \in (a_n, b_n), \\ f(t) & \text{otherwise,} \end{cases}$$

where (a_n, b_n) is an enumeration of all maximal level intervals of f .

Lemma 4.3. Let $f = f^* = \sum_{i=1}^n a_i \chi_{A_i} \in \mathcal{M}_{\varphi, w}$, where $A_i = [t_{i-1}, t_i)$ and $0 = t_0 < t_1 < \dots < t_n < \infty$. Then

$$P_{\varphi, w}(f) = \int_I \varphi\left(\frac{f}{g^f}\right) g^f = \int_I \varphi\left(\frac{f^0}{w}\right) w.$$

In particular, the intervals (t_{i-1}, t_i) are level intervals of f with respect to w . Moreover, the maximal level intervals of f with respect to w are the $(t_{i_j}, t_{i_{j+1}})$, where

$$g^f = \sum_{j=0}^{m-1} \lambda_j f \chi_{[t_{i_j}, t_{i_{j+1}})}$$

is from Algorithm A.

Proof. Let $f = \sum_{i=1}^n a_i \chi_{A_i}$ with $A_i = [t_{i-1}, t_i)$, $0 = t_0 < t_1 < \dots < t_n < \infty$, and

$$g^f = \sum_{j=0}^{m-1} \lambda_j f \chi_{[t_{i_j}, t_{i_{j+1}})}$$

be as in Algorithm A, where $\lambda_0 < \lambda_1 < \dots < \lambda_{m-1}$ and

$$\lambda_j = \frac{W(t_{i_j}, t_{i_{j+1}})}{F(t_{i_j}, t_{i_{j+1}})}, \quad j = 0, 1, \dots, m-1.$$

Hence by Theorem 3.9,

$$(4.1) \quad P_{\varphi, w}(f) = \sum_{j=0}^{m-1} \int_{t_{i_j}}^{t_{i_{j+1}}} \varphi\left(\frac{f}{\lambda_j f}\right) \lambda_j f = \sum_{j=0}^{m-1} \varphi\left(\frac{F(t_{i_j}, t_{i_{j+1}})}{W(t_{i_j}, t_{i_{j+1}})}\right) W(t_{i_j}, t_{i_{j+1}}).$$

We will now compute the level function f^0 with respect to w . Suppose first that

$$w = Tw = \sum_{i=1}^n \left(\frac{1}{|A_i|} \int_{A_i} w \right) \chi_{A_i}.$$

We shall show that every $(t_{i_j}, t_{i_{j+1}})$ is a maximal level interval of f with respect to w . By Remark 4.1 each (t_i, t_{i+1}) is a level interval of f . Moreover one can check that on each (t_i, t_k) , $i < k \leq n$,

$$\forall_{t \in (t_i, t_k)} \quad \frac{F(t_i, t)}{W(t_i, t)} \leq \frac{F(t_i, t_k)}{W(t_i, t_k)} \Leftrightarrow \forall_{i < j < k} \quad \frac{F(t_i, t_j)}{W(t_i, t_j)} \leq \frac{F(t_i, t_k)}{W(t_i, t_k)}.$$

Let us show that each interval $(t_{i_j}, t_{i_{j+1}})$ is a level interval for f with respect to w . In fact we need only to show that

$$R(t_{i_j}, t_k) \leq R(t_{i_j}, t_{i_{j+1}}) \text{ for each } i_j < k < i_{j+1}.$$

Applying the notation of Algorithm A, see (3.8), we have for $j = 0, 1, \dots, m-1$,

$$\lambda_j = \prod_{i=0}^j \gamma_i = \frac{W(t_{i_{j+1}}) - W(t_{i_j})}{F(t_{i_{j+1}}) - F(t_{i_j})} = \frac{1}{R(t_{i_j}, t_{i_{j+1}})},$$

$$g_{j-1} = \lambda_0 f \chi_{[0, t_{i_1})} + \lambda_1 f \chi_{[t_{i_1}, t_{i_2})} + \dots + \lambda_{j-2} f \chi_{[t_{i_{j-2}}, t_{i_{j-1}})} + \lambda_{j-1} f \chi_{[t_{i_{j-1}}, t_n)}.$$

Hence for $i_j < k < i_{j+1}$, $G_{j-1}(t_k) - G_{j-1}(t_{i_j}) = \lambda_{j-1}(F(t_k) - F(t_{i_j}))$, and so

$$\begin{aligned} \frac{1}{R(t_{i_j}, t_{i_{j+1}})} &= \lambda_j = \gamma_j \lambda_{j-1} = \lambda_{j-1} \min_{i_j < i \leq n} \left\{ \frac{W(t_i) - W(t_{i_j})}{G_{j-1}(t_i) - G_{j-1}(t_{i_j})} \right\} \leq \\ &\leq \lambda_{j-1} \frac{W(t_k) - W(t_{i_j})}{G_{j-1}(t_k) - G_{j-1}(t_{i_j})} = \frac{W(t_k) - W(t_{i_j})}{F(t_k) - F(t_{i_j})} = \frac{1}{R(t_{i_j}, t_k)}, \end{aligned}$$

which proves that $(t_{i_j}, t_{i_{j+1}})$ is a level interval.

To see that each $(t_{i_j}, t_{i_{j+1}})$ is a maximal level interval we will need Theorem 3.1 from [4], which states that if $a_1 < a_2 < b_1 < b_2$ and $(a_1, b_1), (a_2, b_2)$ are level intervals of f with respect to w , then (a_1, b_2) is also a level interval of f with respect to w . We also need the simple observation that (a, b) is a level interval if and only if

$$(4.2) \quad R(a, b) \leq R(s, b) \text{ for each } s \in (a, b).$$

The latter is a result of the elementary inequalities that for $v, x, y, z > 0$,

$$\frac{y}{z} \leq \frac{v+y}{x+z} \iff \frac{v+y}{x+z} \leq \frac{v}{x},$$

and that

$$R(a, b) = \frac{F(a, s) + F(s, b)}{W(a, s) + W(s, b)}.$$

Suppose therefore that $(t_{i_j}, t_{i_{j+1}})$ is not maximal. Then there is another level interval (a, b) such that $(t_{i_j}, t_{i_{j+1}}) \subsetneq (a, b)$. It follows that $a < t_{i_j}$ or $t_{i_{j+1}} < b$. Suppose $t_{i_{j+1}} < b$ (in the other case the proof is similar). Then by the mentioned Halperin's result we get that $(t_{i_j}, t_{i_{j+2}})$ is a level interval. But then by definition of level intervals we get

$$\frac{1}{\lambda_j} = R(t_{i_j}, t_{i_{j+1}}) \leq R(t_{i_j}, t_{i_{j+2}}) \leq R(t_{i_{j+1}}, t_{i_{j+2}}) = \frac{1}{\lambda_{j+1}},$$

which means that $\lambda_j \geq \lambda_{j+1}$. However by Algorithm A, $\lambda_{j+1} = \gamma_{j+1} \lambda_j$ with $\gamma_{j+1} > 1$, which gives a contradiction.

Let now w be arbitrary. Denote by $TW(t) = \int_0^t Tw$. Notice that $TW(t_i) = W(t_i)$ for each i , and $TW(t) \leq W(t)$ for any $t > 0$. The latter holds since for any $t \in (t_{k-1}, t_k)$,

$$TW(t) = W(t_{k-1}) + \left(\frac{1}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} w \right) (t - t_{k-1}) \leq W(t_{k-1}) + \int_{t_{k-1}}^t w = W(t).$$

Then for each j and each $t \in (t_{i_j}, t_{i_{j+1}})$ one has $W(t_{i_j}, t) = W(t) - W(t_{i_j}) \geq TW(t) - TW(t_{i_j}) = TW(t_{i_j}, t)$. Therefore, since by the first part of the proof $(t_{i_j}, t_{i_{j+1}})$ is a l.i. of f with respect to Tw ,

$$R(t_{i_j}, t) = \frac{F(t_{i_j}, t)}{W(t_{i_j}, t)} \leq \frac{F(t_{i_j}, t)}{TW(t_{i_j}, t)} \leq \frac{F(t_{i_j}, t_{i_{j+1}})}{TW(t_{i_j}, t_{i_{j+1}})} = \frac{F(t_{i_j}, t_{i_{j+1}})}{W(t_{i_j}, t_{i_{j+1}})} = R(t_{i_j}, t_{i_{j+1}}),$$

which shows that also $(t_{i_j}, t_{i_{j+1}})$ is l.i. of f with respect to w . By the previous reasoning it is also m.l.i. of f with respect to w .

Thus the level function f^0 with respect to w is given by

$$f^0(t) = \begin{cases} R(t_{i_j}, t_{i_{j+1}})w(t) & \text{for } t \in (t_{i_j}, t_{i_{j+1}}), \quad j = 0, 1, \dots, m-1, \\ 0 & \text{for } t \geq t_n. \end{cases}$$

Then, by (4.1),

$$\int_I \varphi(f^0/w)w = \sum_{j=0}^{m-1} \varphi(R(t_{i_j}, t_{i_{j+1}}))W(t_{i_j}, t_{i_{j+1}}) = P_{\varphi, w}(f),$$

and the proof is finished. \square

Remark 4.4. Algorithm A and Lemma 4.3 suggest also another point of view. Namely, rather than changing the function f , we may change the weight according to definition of $P_{\varphi, w}(f)$. Let's define *inverse level function* of w with respect to a decreasing function f as follows

$$w^f(t) = \begin{cases} f(t)/R(a_n, b_n) & \text{for } t \in (a_n, b_n), \\ w(t) & \text{otherwise,} \end{cases}$$

where (a_n, b_n) is an enumeration of all maximal level intervals of f with respect to w . Then by definition of w^f the following equality holds

$$\int_I \varphi\left(\frac{f^0}{w}\right)w = \int_I \varphi\left(\frac{f}{w^f}\right)w^f.$$

Notice also that we have $w^f \prec w$. In fact, for each m.l.i. (a, b) of f with respect to w one has $W(a, b) = W^f(a, b) = \int_a^b w^f$. Moreover, for $t \in (a, b)$,

$$W^f(a, t) = \frac{F(a, t)}{R(a, b)} \leq \frac{F(a, t)}{R(a, t)} = W(a, t).$$

If t is out of any m.l.i. then the equality $W(t) = W^f(t)$ holds. Indeed

$$W^f(t) = \sum_{b_n \leq t} W^f(a_n, b_n) + \int_{E_t} w = \sum_{b_n \leq t} W(a_n, b_n) + \int_{E_t} w = W(t),$$

where $E_t = (0, t) \setminus \bigcup_{b_n \leq t} (a_n, b_n)$. When t is in some m.l.i. then we have only $W^f(t) \leq W(t)$.

The next result is a representation of the modular $P_{\varphi, w}(f)$ via level function of f^* in case when its support is a finite interval.

Proposition 4.5. *Let φ be an N -function. Then for any $f = f^* \in \mathcal{M}_{\varphi, w}$ such that $\text{supp } f = (0, s)$ where $s < \infty$, we have*

$$P_{\varphi, w}(f) = \int_I \varphi\left(\frac{f^0}{w}\right)w = \int_I \varphi\left(\frac{f}{w^f}\right)w^f.$$

Proof. Let $f = f^* \in \mathcal{M}_{\varphi, w}$ and let (C_j) be an enumeration of all m.l.i. of f with respect to w . For every $n \in \mathbb{N}$, let $\mathcal{D}_n = \{(k2^{-n}s, (k+1)2^{-n}s) : 0 \leq k < 2^n\}$ be the set of dyadic subdivisions of the interval $(0, s]$ and $\mathcal{C}_n = \{C_j : j \leq n\}$. The endpoints of all the intervals in $\mathcal{D}_n \cup \mathcal{C}_n$, when rearranged in increasing order, define a finite partition \mathcal{A}_n of the interval $(0, s]$ into subintervals $A_k^n, k = 1, \dots, K(n)$. In other words, \mathcal{A}_n and $\mathcal{D}_n \cup \mathcal{C}_n$ generate the same algebra \mathcal{F}_n of subsets of $(0, s]$. Moreover

$$(4.3) \quad |A_k^n| \leq \frac{s}{2^n}, \quad k = 1, \dots, K(n).$$

Then for each $j \in \mathbb{N}$ and $n \geq j$ there is a finite set $I(j, n) \subset \mathbb{N}$ such that

$$C_j = \bigcup_{k \in I(j, n)} A_k^n.$$

Since $\mathcal{M}_{\varphi,w} \subset L^1 + L^\infty$ [1] and so f is integrable on $[0, s]$, we may define for each n the simple function

$$f_n = \sum_{k=1}^{K(n)} \left(\frac{1}{|A_k^n|} \int_{A_k^n} f \right) \chi_{A_k^n},$$

which is the conditional expectation of f with respect to the algebra \mathcal{F}_n .

We will show that $f_n^0 \rightarrow f^0$ a.e., where f_n^0, f^0 are level functions for f_n, f , respectively. Fix some m.l.i. $C_j = (d, e]$ of f with respect to w . Then $(d, e] = \bigcup_{k \in I(j,n)} A_k^n$ for each $n \geq j$. Thus since f is decreasing and as in Remark 4.1(2), for each $t \in (d, e]$,

$$F_n(d, t) \leq F(d, t), \quad F_n(d, e) = F(d, e).$$

Therefore $(d, e]$ is an l.i. of f_n with respect to w for all $n \geq j$. Clearly for each $n \geq j$, the set C_j is contained in some m.l.i. $C^n = (d_n, e_n]$ of f_n . We claim that

$$(4.4) \quad |C^n - C_j| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

In fact if (4.4) does not hold, there exist a subsequence (n_k) and numbers $d_0, e_0 \in [0, s]$ such that $d_{n_k} \rightarrow d_0$ and $e_{n_k} \rightarrow e_0$, and $d_0 < d$ or $e < e_0$. Moreover $(d_n, e_n]$ is a union of some intervals A_k^n and so $R_n(d_n, e_n) = R(d_n, e_n)$, where R_n is just defined as

$$R_n(s, u) = F_n(s, u)/W(s, u).$$

For each $t \in (d_0, e_0)$ we have that $t \in (d_{n_k}, e_{n_k}]$ for large k . Choose $A_{i(n)}^n = (w_n, v_n]$ and $A_{j(n)}^n = (p_n, r_n]$ in such a way that $t \in A_{i(n)}^n$ and $d_0 \in A_{j(n)}^n$ for each n . If $d_0 < d_{n_k}$, then

$$\begin{aligned} & |F(d_0, t) - F_{n_k}(d_{n_k}, t)| \\ & \leq |F(d_0, d_{n_k})| + |F(d_{n_k}, w_{n_k}) - F_{n_k}(d_{n_k}, w_{n_k})| + |F(w_{n_k}, t) - F_{n_k}(w_{n_k}, t)| \\ & \leq \int_{p_{n_k}}^{d_{n_k}} f + \int_{A_{i(n_k)}^{n_k}} f + \int_{A_{j(n_k)}^{n_k}} f_{n_k} = \int_{p_{n_k}}^{d_{n_k}} f + 2 \int_{A_{i(n_k)}^{n_k}} f. \end{aligned}$$

Similarly, if $d_0 > d_{n_k}$,

$$|F(d_0, t) - F_{n_k}(d_{n_k}, t)| \leq \int_{d_{n_k}}^{r_{n_k}} f + 2 \int_{A_{i(n_k)}^{n_k}} f.$$

In consequence, from both cases in view of (4.3) we get

$$|F(d_0, t) - F_{n_k}(d_{n_k}, t)| \rightarrow 0,$$

and so

$$R(d_0, t) \leftarrow R_{n_k}(d_{n_k}, t) \leq R_{n_k}(d_{n_k}, e_{n_k}) = R(d_{n_k}, e_{n_k}) \rightarrow R(d_0, e_0).$$

It follows that (d_0, e_0) is a l.i. of f , which contradicts our assumption on maximality of C_j and proves (4.4).

Let $t \in C_j$ for some j . Then keeping notation like above we have

$$(4.5) \quad f_n^0(t) = R_n(d_n, e_n)w(t) = R(d_n, e_n)w(t) \rightarrow R(d, e)w(t) = f^0(t).$$

Suppose now $t \in [0, s] \setminus \bigcup_j \overline{C_j}$. Then for all $n \in \mathbb{N}$ there exists $k_0 = k_0(n)$ such that $t \in A_{k_0}^n$. Since $A_{k_0}^n$ are l.i. of f_n , there are m.l.i. M^n of f_n such that $A_{k_0}^n \subset M^n = (m_n, h_n]$. Clearly $(m_n, h_n]$ is a union of some sets A_k^n . One may also explain like in (4.4) that $|M^n| \rightarrow 0$ as $n \rightarrow \infty$, and so for a.a. t ,

$$(4.6) \quad f_n^0(t) = R_n(m_n, h_n)w(t) = R(m_n, h_n)w(t) = \frac{F(m_n, h_n)/|M^n|}{W(m_n, h_n)/|M^n|} w(t) \rightarrow f(t).$$

Thus we get from (4.5) and (4.6) that $f_n^0 \rightarrow f^0$ a.e..

Notice that $P_{\varphi,w}(f_n) \leq P_{\varphi,w}(f)$. In fact, suppose $P_{\varphi,w}(f) = k$. Consider the space $\mathcal{M}_{\psi,w}$, where $\psi(t) = \varphi(t)/k$, with the Luxemburg norm $\|\cdot\|$ given by the modular

$$P_{\psi,w}(f) = \frac{1}{k} P_{\varphi,w}(f).$$

This is a r.i. Banach function space with the Fatou property by Proposition 2.1. Since $f_n \prec f$ we have $\|f_n\|_{\mathcal{M}_{\psi,w}} \leq \|f\|_{\mathcal{M}_{\psi,w}} = 1$ for each n . It follows from the left continuity of the function $(0, \infty) \ni \lambda \mapsto P_{\psi,w}(\lambda f)$ (see Lemma 4.6 in [9]) that $P_{\psi,w}(f_n) \leq 1$, and so

$$P_{\varphi,w}(f_n) \leq P_{\varphi,w}(f).$$

Applying this, the convergence $f_n^0 \rightarrow f^0$ a.e. and $w^f \prec w$ by Remark 4.4, we get

$$\begin{aligned} P_{\varphi,w}(f) &\geq \liminf P_{\varphi,w}(f_n) \stackrel{\text{Lemma 4.3}}{=} \liminf \int_I \varphi(f_n^0/w)w \\ &\stackrel{\text{Fatou Lemma}}{\geq} \int_I \varphi(f^0/w)w = \int_I \varphi(f/w^f)w^f \geq P_{\varphi,w}(f), \end{aligned}$$

which finishes the proof. \square

Lemma 4.6. *Let φ be an N -function and $W(\infty) = \infty$. If $f = f^* \in \mathcal{M}_{\varphi,w}$ then it does not have any degenerate level interval.*

Proof. Suppose there is a degenerate m.l.i. (a, ∞) of f , that is

$$R(a, t) \leq \limsup_{x \rightarrow \infty} R(a, x) = R(a, \infty) \text{ for each } t > a, \text{ and } R(a, \infty) > 0.$$

Without loss of generality we also suppose that $P_{\varphi,w}(f) < \infty$. We can do this since level intervals of f are the same for all kf , where $k > 0$.

We will consider three cases.

a) Suppose $R(a, t) < \limsup_{x \rightarrow \infty} R(a, x)$ for each $t > a$. Define

$$x_n = \max\{x \in [a, a+n] : R(a, x) = \sup\{R(a, t) : t \in [a, a+n]\}\}.$$

We have that $x_n \nearrow \infty$ and $R(a, \infty) = \lim_{n \rightarrow \infty} R(a, x_n)$. In fact if $x_n \rightarrow x_0 < \infty$ then by the assumption $R(a, x_0) = \lim_{n \rightarrow \infty} R(a, x_n) = \sup_{t \in (a, \infty)} R(a, t) < R(a, \infty)$, which is impossible. Therefore $x_n \nearrow \infty$ and $\lim_{n \rightarrow \infty} R(a, x_n) = \sup_{t \in (a, \infty)} R(a, t) = \limsup_{t \rightarrow \infty} R(a, t) = R(a, \infty)$.

Consider the sequence of functions $g_n = f\chi_{(0, x_n]}$. Clearly $R(a, t) \leq R(a, x_n)$ for each $a < t < x_n$. Hence $(a, x_n]$ is an l.i. of f and thus it is an m.l.i. of g_n . Therefore $g_n^0 = f^0\chi_{(0, a)} + R(a, x_n)w\chi_{[a, x_n]} \rightarrow f^0\chi_{(0, a)} + R(a, \infty)w\chi_{[a, \infty)} = f^0$, and by Proposition 4.5 applied to g_n we have

$$\begin{aligned} P_{\varphi,w}(g_n) &= \int_0^{x_n} \varphi(g_n^0/w)w = \\ &= \int_0^a \varphi(f^0/w)w + \int_a^{x_n} \varphi(R(a, x_n))w \\ &= \int_0^a \varphi(f^0/w)w + \varphi(R(a, x_n))(W(x_n) - W(a)) \rightarrow \infty, \end{aligned}$$

since $\int_a^\infty w = \infty$ by the assumption $W(\infty) = \infty$. On the other hand $P_{\varphi,w}(g_n) \leq P_{\varphi,w}(f)$ and so $P_{\varphi,w}(f) = \infty$, which is a contradiction to our assumption.

Consider now the following set

$$B = \{z > a : R(a, z) = R(a, \infty)\}.$$

If the case a) is not satisfied then $B \neq \emptyset$.

b) Let first $\sup B = \infty$. Then there exists $a < x_n \nearrow \infty$ such that $R(a, x_n) = R(a, \infty)$ for each $n \in \mathbb{N}$, and we proceed as in a).

c) Suppose now that $\sup B = b < \infty$. Clearly $R(a, b) = R(a, \infty)$. Let $b < y_n \nearrow \infty$ be such that $R(a, y_n) \nearrow R(a, \infty)$. Then for each $\sigma > 1$ there exists N such that for $n > N$ we have

$$R(a, y_n) \leq R(a, b) \leq \sigma R(a, y_n).$$

We will show that for sufficiently large n ,

$$(4.7) \quad R(b, y_n) \leq R(a, y_n) \leq \sigma R(b, y_n).$$

The left side of this inequality follows immediately from (4.2). In order to get the right side notice first that

$$\frac{F(a, b)}{W(a, b)} = R(a, b) \leq \sigma R(a, y_n) = \sigma \frac{F(a, b) + F(b, y_n)}{W(a, b) + W(b, y_n)}.$$

Then

$$F(a, b)W(b, y_n) \leq \sigma F(b, y_n)W(a, b) + (\sigma - 1)F(a, b)W(a, b),$$

and since $W(b, y_n) \rightarrow \infty$,

$$F(a, b)W(b, y_n) \leq \sigma F(b, y_n)W(a, b) + (\sigma - 1)F(b, y_n)W(b, y_n)$$

for n large enough. Hence

$$F(a, b)W(b, y_n) + F(b, y_n)W(b, y_n) \leq \sigma[F(b, y_n)W(a, b) + F(b, y_n)W(b, y_n)]$$

and so

$$R(a, y_n) = \frac{F(a, b) + F(b, y_n)}{W(a, b) + W(b, y_n)} \leq \sigma \frac{F(b, y_n)}{W(b, y_n)} = \sigma R(b, y_n),$$

and the inequality (4.7) is proved.

Therefore $R(b, y_n) \rightarrow R(a, b) = R(a, \infty) = R(b, \infty)$. Moreover, once again using (4.2) for each $b < t$ from $R(a, t) < R(a, b)$ we have

$$R(b, t) < R(a, t) < R(a, b) = R(a, \infty) = R(b, \infty),$$

where the second inequality follows from definition of B . Therefore (b, ∞) is an l.i. of f . Notice also that (b, ∞) is of the same type as the interval (a, ∞) in the case a). Choosing (x_n) like in that case for b instead of a we define $g_n = f\chi_{[0, x_n]}$. Then m.l.i. of g_n are the same like for f in the interval $[0, a]$. Moreover, by the assumption $\sup B = b < \infty$, the interval $(a, b]$ is an m.l.i. of g_n , and by definition of (x_n) , the interval $(b, x_n]$ is an m.l.i. of g_n . Hence the sequence (g_n) is increasing and

$$g_n^0 = f^0\chi_{(0, a]} + R(a, b)w\chi_{(a, b]} + R(b, x_n)w\chi_{(b, x_n]}.$$

Thus

$$g_n \nearrow f^0\chi_{(0, a]} + R(a, b)w\chi_{(a, b]} + R(b, \infty)w\chi_{(b, \infty)} = f^0 \text{ a.e.,}$$

and we conclude as in a). The proof is completed. \square

Now we state the main theorem of this section.

Theorem 4.7. *Let φ be an N -function and $W(\infty) = \infty$. Then for any $f = f^* \in \mathcal{M}_{\varphi, w}$ we have*

$$P_{\varphi, w}(f) = \int_I \varphi\left(\frac{f^0}{w}\right)w = \int_I \varphi\left(\frac{f}{w^f}\right)w^f.$$

Proof. Let $f = f^* \in \mathcal{M}_{\varphi, w}$ be arbitrary with $P_{\varphi, w}(f) < \infty$. In view of Proposition 4.5 we assume that $\text{supp } f = (0, \infty)$. Applying Lemma 4.6, f does not have any degenerate level interval. Thus it remains to consider the following two cases.

First suppose there is a sequence $s_n \nearrow \infty$ such that each s_n is on the boundary of some m.l.i. of f . Define $g_n = f\chi_{[0, s_n]}$. Then $g_n \nearrow f$ a.e. and by Lemma 4.6 in [9],

$$P_{\varphi, w}(g_n) \rightarrow P_{\varphi, w}(f).$$

Moreover, for such chosen (s_n) each m.l.i. of g_n is also m.l.i. of f and therefore we see that $g_n^0 = f^0 \chi_{[0, s_n]}$. Then $g_n^0 \nearrow f^0$ and by Proposition 4.5 and the Lebesgue Convergence Theorem,

$$P_{\varphi, w}(g_n) = \int_I \varphi(g_n^0/w)w \rightarrow \int_I \varphi(f^0/w)w,$$

which gives the claim.

Now assume there is s such that each l.i. of f is in $[0, s]$. Take (s_n) satisfying $s < s_n \nearrow \infty$ and put $g_n = f \chi_{[0, s_n]}$. Then once again $g_n^0 = f^0 \chi_{[0, s_n]}$, because there is no l.i. of g_n in (s, s_n) , and we conclude this case as above. The proof is completed. \square

Summarizing main results of sections 2 and 4 (especially Theorems 2.2 and 4.7) we get the following theorem.

Theorem 4.8. *Let w be a decreasing weight and φ be N -function. Then the Köthe dual spaces to Orlicz-Lorentz spaces $\Lambda_{\varphi, w}$ and $\Lambda_{\varphi, w}^0$ are expressed as*

$$(\Lambda_{\varphi, w})' = \mathcal{M}_{\varphi_*, w}^0 \quad \text{and} \quad (\Lambda_{\varphi, w}^0)' = \mathcal{M}_{\varphi_*, w},$$

with

$$\begin{aligned} \|f\|_{(\Lambda_{\varphi, w})'} &= \|f\|_{\mathcal{M}_{\varphi_*, w}^0} = \inf_{k>0} \left\{ \frac{1}{k} (P_{\varphi_*, w}(kf) + 1) \right\}, \\ \|f\|_{(\Lambda_{\varphi, w}^0)'} &= \|f\|_{\mathcal{M}_{\varphi_*, w}} = \inf \{ \lambda > 0 : P_{\varphi_*, w}(f/\lambda) \leq 1 \}, \end{aligned}$$

where

$$P_{\varphi_*, w}(f) = \inf \left\{ \int_I \varphi_*(f^*/|g|) |g| : g \prec w \right\}.$$

If in addition we assume that $W(\infty) = \infty$ for $I = [0, \infty)$ then we also have that

$$P_{\varphi_*, w}(f) = \int_I \varphi_*((f^*)^0/w)w = \int_I \varphi_*(f^*/w^{f^*})w^{f^*},$$

where $(f^*)^0$ is a level functions of f^* with respect to w and w^{f^*} is an inverse level function of w with respect to f^* .

For $\varphi(u) = \frac{1}{p}u^p$, $1 < p < \infty$, we denote the space $\Lambda_{\varphi, w}$ by $\Lambda_{p, w}$. The next corollary provides an isometric description of $(\Lambda_{p, w})'$. The second formula recovers Halperin's Theorem 6.1 and Corollary on page 288 in [4].

Corollary 4.9. *Let $1 < p < \infty$ and $1/p + 1/q = 1$. Then for any $f \in (\Lambda_{p, w})'$ we have*

$$\|f\|_{(\Lambda_{p, w})'} = \inf \left\{ \left(\int_I (f^*/|g|)^q |g| \right)^{1/q} : g \prec w \right\}.$$

If in addition $W(\infty) = \infty$ in case of $I = [0, \infty)$, then

$$\|f\|_{(\Lambda_{p, w})'} = \left(\int_I ((f^*)^0/w)^q w \right)^{1/q}.$$

Proof. The first equality follows from Theorem 2.2, while the second one from Theorem 4.7. \square

Remark 4.10. In Lorentz's paper [12] a theorem (Theorem 3.6.5) on duality of the space $\Lambda_{p, w}$ for $1 < p < \infty$ was stated in terms of "level functions", however his definition of a level function is different from the one introduced earlier by Halperin. A similar notion of a level function has been later used by Sinnamoni (see [23], Chapter 2.9). Both Lorentz's and Halperin's representations suggest that $f^0/w = (f/w)^L$ for every non-negative and decreasing function f , where the right side means the level function of f/w in the Lorentz sense. It is straightforward to check this equality for a characteristic decreasing function.

5. SEQUENCE CASE.

We complete the discussion on the dual of Orlicz-Lorentz spaces considering here the discrete case. All results above for function spaces are valid in the Orlicz-Lorentz sequence spaces as well. Recall that for a given sequence $x = (x_i)$, its decreasing rearrangement $x^* = (x_i^*)$ is defined as $x_i^* = \inf \{ \lambda > 0 : d_x(\lambda) < i \}$, $i \in \mathbb{N}$, where $d_x(\lambda) = \mu \{ i \in \mathbb{N} : |x_i| > \lambda \}$ for $\lambda > 0$, and μ is a counting measure. Then given an Orlicz function φ and a decreasing positive weight sequence $w = (w_i)$, the Orlicz-Lorentz sequence space $\lambda_{\varphi, w}$ is defined as

$$\lambda_{\varphi, w} = \left\{ x = (x_i) \in l^0 : \exists_{\delta > 0} \sum_{i=1}^{\infty} \varphi(\delta x_i^*) w_i < \infty \right\},$$

where l^0 is the space of real valued sequences. We consider the space $\lambda_{\varphi, w}$ with the Luxemburg norm $\| \cdot \|_{\lambda_{\varphi, w}}$ denoted further by $\lambda_{\varphi, w}$, or with the Amemiya norm $\| \cdot \|_{\lambda_{\varphi, w}}^0$ denoted by $\lambda_{\varphi, w}^0$. Those norms are defined analogously as for function spaces. The Orlicz-Lorentz sequence spaces are Köthe spaces as subspaces of l^0 , and their Köthe dual spaces are defined analogously as in function case. For each $x \in \lambda_{\varphi, w}$ we assign an element $\bar{x} \in \Lambda_{\varphi, \bar{w}}$ on $[0, \infty)$, where

$$\bar{x} = \sum_{i=1}^{\infty} x_i \chi_{[i-1, i)} \quad \text{and} \quad \bar{w} = \sum_{i=1}^{\infty} w_i \chi_{[i-1, i)}.$$

The above correspondence between x and \bar{x} is a linear isometry between $\lambda_{\varphi, w}$ and a closed subspace of $\Lambda_{\varphi, \bar{w}}$. Evidently

$$\|x\|_{\lambda_{\varphi, w}} = \|\bar{x}\|_{\Lambda_{\varphi, \bar{w}}} \quad \text{and} \quad \|x\|_{\lambda_{\varphi, w}}^0 = \|\bar{x}\|_{\Lambda_{\varphi, \bar{w}}}^0.$$

The lemma below ensures that the respective correspondence remains true in the dual space.

Lemma 5.1. *Let $y = (y_i) \in (\lambda_{\varphi, w})'$. Then*

$$\|y\|_{(\lambda_{\varphi, w})'} = \|\bar{y}\|_{(\Lambda_{\varphi, \bar{w}})'} \quad \text{and} \quad \|y\|_{(\lambda_{\varphi, w}^0)'} = \|\bar{y}\|_{(\Lambda_{\varphi, \bar{w}}^0)' }.$$

Proof. Define an averaging operator T on $\Lambda_{\varphi, \bar{w}}$ by

$$T : h \rightarrow \sum_{i=1}^{\infty} \left(\int_{[i-1, i)} h \right) \chi_{[i-1, i)}.$$

Then by [1, Theorem 4.8], $\|Th\|_{\Lambda_{\varphi, \bar{w}}} \leq \|h\|_{\Lambda_{\varphi, \bar{w}}}$ for each $h \in \Lambda_{\varphi, \bar{w}}$. Moreover, for any $y \in (\lambda_{\varphi, w})'$,

$$\int_0^{\infty} \bar{y} h = \int_0^{\infty} \bar{y} (Th).$$

Therefore

$$\begin{aligned} \|\bar{y}\|_{(\Lambda_{\varphi, \bar{w}})'} &= \sup \left\{ \int_0^{\infty} \bar{y} h : \|h\|_{\Lambda_{\varphi, \bar{w}}} \leq 1 \right\} = \sup \left\{ \int_0^{\infty} \bar{y} (Th) : \|h\|_{\Lambda_{\varphi, \bar{w}}} \leq 1 \right\} \\ &= \sup \left\{ \int_0^{\infty} \bar{y} (Th) : \|Th\|_{\Lambda_{\varphi, \bar{w}}} \leq 1 \right\} = \sup \left\{ \int_0^{\infty} \bar{y} z : \|z\|_{\lambda_{\varphi, w}} \leq 1 \right\} \\ &= \sup \left\{ \sum_{i=1}^{\infty} y_i z_i : \|z\|_{\lambda_{\varphi, w}} \leq 1 \right\} = \|y\|_{(\lambda_{\varphi, w})'}. \end{aligned}$$

Similarly we prove the second equality. □

By analogy to function case the following space has been defined in [9],

$$\mathfrak{m}_{\varphi, w} = \{ x \in l^0 : \exists_{\lambda > 0} p_{\varphi, w}(x/\lambda) < \infty \},$$

with the modular

$$p_{\varphi,w}(x) = \inf \left\{ \sum_{i=1}^{\infty} \varphi(x_i^*/|y_i|)|y_i| : y \prec w \right\},$$

where the submajorization of sequences $y \prec w$ means that $\sum_{i=1}^n y_i^* \leq \sum_{i=1}^n w_i$ for all $n \in \mathbb{N}$. By $\mathbf{m}_{\varphi,w}$ denote the space equipped with the Luxemburg norm $\|\cdot\|_{\mathbf{m}}$ and by $\mathbf{m}_{\varphi,w}^0$ the space endowed with the Amemiya norm $\|\cdot\|_{\mathbf{m}}^0$. Moreover, we adopt definitions of the previous chapter to the sequence case setting for decreasing sequence $x = (x_i)$ and $a, b \in \mathbb{N} \cup \{0\}$, $a < b$,

$$w(a, b) = \sum_{i=a+1}^b w_i, \quad x(a, b) = \sum_{i=a+1}^b x_i, \quad r(a, b) = \frac{x(a, b)}{w(a, b)}.$$

Then $(a, b] = \{a+1, \dots, b\} \subset \mathbb{N}$ is called a *level interval of x with respect to w* if for each $j = a+1, \dots, b$,

$$r(a, j) \leq r(a, b) \text{ and } 0 < r(a, b),$$

and the *level sequence x^0 of x with respect to w* is defined as

$$x_i^0 = \begin{cases} r(a_n, b_n) w_i & \text{for } i \in (a_n, b_n], \\ x_i & \text{otherwise,} \end{cases}$$

where $(a_n, b_n]$ is a sequence of all maximal level intervals of x . Notice that the results of the previous section ensure that the correspondence between x and \bar{x} preserves such defined level intervals. In fact (see the proofs of Lemmas 3.2, 4.3) we have for any $a \in \mathbb{N} \cup \{0\}$, $r(a, j) \leq r(a, b)$ for all $j = a+1, \dots, b$, if and only if $\bar{x}(a, t)/\bar{w}(a, t) \leq \bar{x}(a, b)/\bar{w}(a, b)$ for all $t \in (a, b)$. Hence $(a, b] \subset \mathbb{N}$ is an m.l.i. of x with respect to w if and only if (a, b) is an m.l.i. of \bar{x} with respect to \bar{w} . Therefore

$$(5.1) \quad \int_0^\infty \varphi((\bar{x}^*)^0/\bar{w})\bar{w} = \sum_{i=1}^\infty \varphi((x_i^*)^0/w_i)w_i.$$

Moreover, in view of Lemma 3.2, $y \prec w$ if and only if $\bar{y} \prec \bar{w}$ and thus

$$p_{\varphi,w}(x) = \inf \left\{ \int_0^\infty \varphi(\bar{x}^*/|\bar{y}|)|\bar{y}| : \bar{y} \prec \bar{w} \right\}.$$

Hence by Lemma 3.1 applied to the step function \bar{x}^* we obtain that

$$(5.2) \quad P_{\varphi,\bar{w}}(\bar{x}) = p_{\varphi,w}(x).$$

Finally, employing equalities (5.1), (5.2), Lemma 5.1 and Theorem 4.8, we can state duality result for Orlicz-Lorentz sequence space $\lambda_{\varphi,w}$.

Theorem 5.2. *Let w be a decreasing weight sequence and φ be an N -function. Then*

$$(\lambda_{\varphi,w})' = \mathbf{m}_{\varphi^*,w}^0 \text{ and } (\lambda_{\varphi,w}^0)' = \mathbf{m}_{\varphi^*,w}.$$

If in addition $\sum_{i=1}^\infty w_i = \infty$, then

$$p_{\varphi^*,w}(x) = \sum_{i=1}^\infty \varphi_*\left(\frac{(x_i^*)^0}{w_i}\right)w_i.$$

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